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On the derivation of the Navier–Stokes–alpha equations from Hamilton's principle

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We investigate the derivation of Euler's equation from Hamilton's variational principle for flows decomposed into their mean and fluctuating parts. Our particular concern is with the flow decomposition used in the derivation of the Navier–Stokes–α equation which expresses the fluctuating velocity in terms of the mean flow and a small fluctuating displacement. In the past the derivation has retained terms up to second order in the Lagrangian which is then averaged. The variation is effected by incrementing the mean velocity, while holding the moments of the products of the displacements fixed. The process leads to a mean Euler equation for the mean velocity. The Navier–Stokes–α equation is only obtained after making a further closure approximation, which is not the concern of this paper. Instead attention is restricted here to the exact analysis of Euler's equation. We show that a proper implementation of Hamilton's principle, which concerns the virtual variation of particle paths, can only be achieved when the fluctuating displacement and mean velocity are varied in concert. This leads to an exact form of Euler's equation. If, on the other hand, the displacement is held fixed under the variation, a term in Euler's equation is lost. Averaging that erroneous form provides the basis of the Navier–Stokes–α equation. We explore the implications of the correct mean equation, particularly with regard to Kelvin's circulation theorem, comparing it with the so called GLM and glm-equations.

1. Introduction

Increasing attention has been paid during the past decade to a regularized form of the Euler equations that was first postulated by Holm, Marsden & Ratiu (1998) and that builds on the earlier Camassa–Holm equation for one-dimensional compressible flow. The regularized equations, originally named after Camassa & Holm (1993), are now more generally known as 'the Navier–Stokes–α equations', or simply 'NS–α equations'. The theme has been developed further in a series of papers (Holm 1999, 2002*b*; Marsden & Shkoller 2003). Despite its name, viscous forces are omitted from the NS–α equations, and they will therefore be called here 'the Euler–α equations'. A generalized Euler–α model has also been proposed that includes viscosity; see for example Foias, Holm & Titi (2001). This is variously called 'the viscous Camassa–Holm model' and 'the Kelvin–filtered Navier–Stokes equation' and has been applied to turbulence. In fact, even the Euler–α theory has proved its worth in parameterizing unresolved 'subgrid scales' in numerical simulations of systems having a wide range of length scales. It also has, in the case of incompressible or barotropic fluids, the attractive property of preserving Kelvin's theorem on the constancy of fluid circulation round any closed circuit moving with the fluid. The model has been generalized to other contexts, such as magnetohydrodynamic flows (see for example Holm 2002*a*; Roberts & Soward 2006).

As explained by Holm (2002*b*), the Euler–α equations have so far been derived only from Hamilton's principle. His strategy was to decompose the velocity into a mean and fluctuating part, which itself is represented by a fluctuating displacement advected by the mean flow. The Lagrangian for the application of the variational method is averaged, neglecting all terms smaller than quadratic in the fluctuating displacement. Normally, in the application of Hamilton's principle, the virtual change in the velocity is equal to the virtual change in either the mean or fluctuating velocity, whichever is varied. In Holm's approach, the fluctuating velocity is a function of the mean velocity, because the fluctuating displacement is advected by it. So under a virtual change of the mean velocity, the fluctuating velocity varies in concert. Thus the virtual change of the velocity is not achieved by a virtual change of the mean velocity alone, as envisaged by Holm (2002*b*), but must be accompanied by a corresponding change in the virtual displacement. Owing to this oversight, terms are lost from Holm's averaged Euler equation. This loss appears to conflict with the claim made by Gjaja & Holm (1996) that the order in which averaging the Lagrangian and implementing the virtual displacement are undertaken is immaterial.

Although our conclusions generalize easily, we shall focus only on the Euler–α equations for the motion of an inviscid non-rotating non-magnetic incompressible uniform-density fluid. We provide the correct form of the averaged Euler equation and derive it directly from Hamilton's principle by taking proper account of the relation between the mean velocity and the fluctuating displacement. The omission of terms in Holm's mean equation is serious because it leads to an averaged version of Kelvin's Theorem stating that the circulation round a closed loop moving with the mean velocity is conserved. This attractive property of his equations is lost when the omitted terms are restored.

Our paper is organized as follows. In §2, we sketch the classical derivation of Euler's equation from Hamilton's principle first, in §2.1, from the natural particle-path (Lagrangian) point of view, while, in §2.2, we provide the appropriate extensions necessary to derive the result within the Eulerian framework. This review provides a reference for our later refinements. In §3, we decompose the flow in two ways: One is a hybrid Eulerian–Lagrangian (HEL) description discussed in §3.1, which builds on the results of §2.1 and the other, described in §3.2, extends the Eulerian description of §2.2. At this stage the basis of the decomposition is arbitrary so as to emphasize its general structure. However when we take averages in §4, the decompositions are defined on the basis of mean and fluctuating velocities, following Andrews & McIntyre (1978*a,b*) in the HEL case of §3.1, and following Holm (2002*b*) in the Eulerian case of §3.2. The various versions of the mean Euler equation are compared, together with their consequences for the evolution of mean circulation, in §4. We are only able to make quantitative comparisons between the HEL and Eulerian mean equations in the small-displacement limit, which we consider in §5. We add a few concluding remarks in §6, where we highlight our principal results.

Unfortunately, we have needed to use 'Lagrangian' in two different senses: one in the sense of a Lagrange density in the variational calculus; the other in connection with the particle-path description of the flow. In what follows, the appropriate meaning should be self-evident from the context in which 'Lagrange' and 'Lagrangian' are used.

2. Euler's equation from Hamilton's principle

Euler's equation may be expressed succinctly in the form

$$\mathscr{D}_t\, \boldsymbol{v} + \rho^{-1}\nabla p = \mathbf{0}\,, \tag{2.1a}$$

where $\boldsymbol{v}$, ρ and p are the velocity, density and pressure respectively at position $\boldsymbol{x}$ at time t and

$$\mathscr{D}_t \;\equiv\; \partial_t \;+\; \boldsymbol{v}\boldsymbol{\cdot}\nabla \tag{2.1b}$$

is the material derivative. Throughout this paper we will be concerned with fluids of constant density $\rho = \rho_0$, for which the flow velocity is solenoidal

$$\nabla\boldsymbol{\cdot}\boldsymbol{v} = 0 \tag{2.2}$$

and Euler's equation (2.1a) may then be expressed conveniently in the form

$$\boldsymbol{\mathscr{E}} \;=\; \mathbf{0}\,, \qquad \text{where} \qquad \boldsymbol{\mathscr{E}} \;\equiv\; \mathscr{D}_t\, \boldsymbol{v} \;+\; \nabla(p/\rho_0)\,. \tag{2.3a,b}$$

In this section we briefly review the derivation of Euler's equation (2.3a) from Hamilton's principle. For an introduction to Hamilton's principle as applied to continuous systems, see for example Chapter 12 of Goldstein (1980). We begin with the classical variational approach based on the Lagrangian flow description in §2.1 and describe the modified approach based on the Eulerian flow description in §2.2. The techniques described here are then adapted to accommodate alternative hybrid flow descriptions in §3.

2.1. *The Lagrangian description*

As is well known, the Lagrangian flow description involves tracking a fluid particle as it moves from some initial point $\boldsymbol{a}$, at time $t = t_0$, to its position, $\boldsymbol{x}^\ell = \boldsymbol{x}^\ell(\boldsymbol{a}, t)$, at time t. We identify the value of a function, $\phi(\boldsymbol{x}, t)$ (say), at this moving point by the addition of the superscript ℓ, and write $\phi^\ell(\boldsymbol{a}, t) = \phi(\boldsymbol{x}^\ell(\boldsymbol{a}, t), t)$. We call $\boldsymbol{a}$ and $\boldsymbol{x}^\ell$ the Lagrangian and Eulerian coordinates respectively. At various stages in this paper, we wish to use one of the two vectors $\boldsymbol{x}^\ell$ or $\boldsymbol{a}$ as the independent variable and then call it $\boldsymbol{x}$. So, for example, in (2.1)–(2.3) the independent variable is the Eulerian coordinate $\boldsymbol{x}\,(= \boldsymbol{x}^\ell)$, while in this subsection the independent variable is the Lagrangian coordinate $\boldsymbol{x}\,(= \boldsymbol{a})$ upon which the Eulerian coordinate depends: $\boldsymbol{x}^\ell = \boldsymbol{x}^\ell(\boldsymbol{x}, t)$. From this point of view, the addition of the superscript ℓ to the function ϕ is an 'operation' that determines

$$\phi^\ell(\boldsymbol{x}, t) \;\equiv\; \phi(\boldsymbol{x}^\ell(\boldsymbol{x}, t), t)\,, \tag{2.4}$$

i.e. though the argument of ϕ^ℓ is the Lagrangian coordinate $\boldsymbol{x}$ (= $\boldsymbol{a}$), it determines the value of ϕ elsewhere at the Eulerian position $\boldsymbol{x}^\ell(\boldsymbol{x}, t)$.

Though differentiation of $\phi^\ell(\boldsymbol{x}, t)$ is straightforward, we should be clear about what happens and how the ℓ–operator notation is used. To begin we note that its gradient is

$$\frac{\partial \phi^\ell}{\partial x_i} \;=\; \frac{\partial x_j^\ell}{\partial x_i}\left(\frac{\partial \phi}{\partial x_j}\right)^{\ell} \qquad \text{implying} \qquad \nabla^\ell \phi^\ell \;=\; (\nabla\phi)^\ell\,, \tag{2.5a,b}$$

since $(\nabla^\ell\phi^\ell)_i \;\equiv\; \partial\phi^\ell/\partial x_i^\ell \;=\; (\partial x_k/\partial x_i^\ell)(\partial\phi^\ell/\partial x_k)$. In turn this result may be used to establish that the rate of change of ϕ for an individual fluid particle is

$$\partial_t(\phi^\ell) = (\mathscr{D}_t\phi)^\ell \qquad \text{with} \qquad \boldsymbol{v}^\ell \;=\; \partial_t(\boldsymbol{x}^\ell) = (\mathscr{D}_t\boldsymbol{x})^\ell\,. \tag{2.6a,b}$$

With this notation Euler's equation (2.1a) at the moving point $\boldsymbol{x}^\ell(\boldsymbol{x}, t)$ becomes

$$(\mathscr{D}_t\, \boldsymbol{v} \;+\; \rho^{-1}\nabla p)^\ell \;=\; \partial_t(\boldsymbol{v}^\ell) \;+\; (\rho^\ell)^{-1}\nabla^\ell p^\ell \;=\; \mathbf{0}\,. \tag{2.7}$$

Though this form of Euler's equation looks unfamiliar, it serves to emphasize that ∂_t in this context is the material derivative following the moving point $\boldsymbol{x}^\ell$.

Just as gradients with respect to $\boldsymbol{x}$ and $\boldsymbol{x}^\ell$ are linked by (2.5*a*), line elements are linked according to the relation $\mathrm{d}x_i^\ell = (\partial x_i^\ell/\partial x_j)\,\mathrm{d}x_j$ and, in consequence, volume elements by $\mathrm{d}^3x^\ell = \mathscr{J}\mathrm{d}^3x$, where $\mathscr{J}(\boldsymbol{x},t) = \det(\partial x_i^\ell/\partial x_j)$ is the Jacobian of the transformation. Consider a material volume element d^3x, located at the initial position $\boldsymbol{x} = \boldsymbol{a}$ at time t_0. It has mass $\rho_0\mathrm{d}^3x$, where $\rho_0 =$ constant. Subsequently for $t > t_0$ the volume element moves with the fluid occupying the volume d^3x^ℓ at its current position $\boldsymbol{x}^\ell(\boldsymbol{x},t)$, where its mass has become $\rho^\ell\mathrm{d}^3x^\ell$. Since mass is conserved it follows that

$$\rho^\ell\mathrm{d}^3x^\ell = \rho_0\,\mathrm{d}^3x \qquad \text{implying} \qquad \mathscr{J}\rho^\ell = \rho_0\,. \tag{2.8a,b}$$

In anticipation of our variational derivation of Euler's equation for a constant-density fluid from Hamilton's principle, we remark that we wish to find the stationary value of a space–time integral of the kinetic energy density $\frac{1}{2}\rho\,|\boldsymbol{v}|^2$ under variations of $\boldsymbol{v}$ subject to the constraint of incompressibility. Following standard practice for such variational problems, we permit compressible variations but instead impose the incompressibility constraint via the Lagrange multiplier method. To that end we consider the Lagrange density

$$L(\boldsymbol{v},\rho,p) = \tfrac{1}{2}\rho\,|\boldsymbol{v}|^2 - p\,((\rho/\rho_0) - 1), \tag{2.9a}$$

in which p is introduced as a Lagrange multiplier but will turn out to be the pressure appearing in Euler's equation (2.1*a*). Variation with respect to p will recover $\rho = \rho_0$. Until we do that, however, ρ like $\boldsymbol{v}$ and p must be regarded as a function of $\boldsymbol{x}$ and t. We note that, although alternative Lagrange multiplier formulations are possible involving the constraint $\nabla\boldsymbol{\cdot}\boldsymbol{v} = 0$ (see (2.2)) instead of $\rho - \rho_0 = 0$, the choice (2.9*a*) appears to be the most convenient form for our later developments in §3. This is because, if we rewrite (2.9*a*) in the form

$$L(\boldsymbol{v},\rho,p) = p - \rho\,\Pi\,, \qquad \text{where} \qquad \Pi = (p/\rho_0) - \tfrac{1}{2}|\boldsymbol{v}|^2, \tag{2.9b,c}$$

ρ variations lead naturally to a term $\nabla\Pi$, which is more fundamental than ∇p in the hybrid Eulerian–Lagrangian developments of §3.1.

Following Goldstein (1980), the current state of a continuous system can be described by a 'point in an infinite-dimensional parameter space'. The parameters are an infinite set of particle labels, for which we are here using their Lagrangian coordinates $\boldsymbol{a}$, though other choices are envisaged later. The evolution of the system from an initially specified state at an initial time t_0 to the state at the final time t_1 can be visualized as a 'trajectory in this infinite-dimensional parameter space'. Hamilton's principle is concerned with the Action integral

$$\mathscr{A} = \int_{t_0}^{t_1}\mathscr{L}\,\mathrm{d}t\,, \qquad \text{where} \qquad \mathscr{L} = \int_{\mathscr{V}_0}\mathscr{J}L^\ell\,\mathrm{d}^3x = \int_{\mathscr{V}^\ell}L^\ell\,\mathrm{d}^3x^\ell \tag{2.10a,b}$$

is the Lagrangian of the system and L^ℓ is the Lagrange density

$$L^\ell = L(\boldsymbol{v}^\ell,\rho^\ell,p^\ell), \qquad \text{where} \qquad \mathscr{J}L^\ell = \tfrac{1}{2}\rho_0|\boldsymbol{v}^\ell|^2 - p^\ell(1-\mathscr{J}) \tag{2.11a,b}$$

(see (2.8*b*) and (2.9*a*)). The independent variables in the first integral $\int_{\mathscr{V}_0}\mathscr{J}L^\ell\,\mathrm{d}^3x$ of (2.10*b*) and second integral $\int_{\mathscr{V}^\ell}L^\ell\,\mathrm{d}^3x^\ell$ are the Lagrangian coordinate $\boldsymbol{x}\,(=\boldsymbol{a}) \in \mathscr{V}_0$ and the Eulerian coordinate $\boldsymbol{x}^\ell \in \mathscr{V}^\ell$ respectively, where $\mathscr{V}_0$ and $\mathscr{V}^\ell$ are the initial $t = t_0$ and current $t_0 \leqslant t \leqslant t_1$ volumes occupied by the fluid, i.e. the volume mapping $\mathscr{V}_0 \mapsto \mathscr{V}^\ell$ corresponds to the point mapping $\boldsymbol{x} \mapsto \boldsymbol{x}^\ell$. Hamilton's principle compares the value of the action integral $\mathscr{A}$ for the actual trajectory $\boldsymbol{x}^{\ell}(\boldsymbol{x},t)$ with the action integral for all other neighbouring 'virtual trajectories', which are displaced from the actual trajectory, the positions of the particles being $\boldsymbol{x}^{\ell}(\boldsymbol{x},t)+\boldsymbol{\eta}^{\ell}(\boldsymbol{x},t)$ where $\boldsymbol{\eta}^{\ell}(\boldsymbol{x},t_0) = \boldsymbol{\eta}^{\ell}(\boldsymbol{x},t_1) = \boldsymbol{0}$; it is also required that the normal component of $\boldsymbol{\eta}^{\ell}(\boldsymbol{x},t)$ vanishes on the boundary of $\mathscr{V}^{\ell}$ for all t. A virtual trajectory is said to be neighbouring when $\boldsymbol{\eta}^{\ell}(\boldsymbol{x},t)$ is infinitesimal. The dynamical equations are not in general satisfied on a virtual trajectory. Hamilton's principle is a variational statement that picks out the trajectory on which the dynamics are satisfied. It states that the action integral $\mathscr{A}$, also simply called 'the action', for the actual trajectory takes a stationary value compared with the action $\mathscr{A}+\delta\mathscr{A}$ for all neighbouring virtual trajectories.

Though the virtual displacement $\boldsymbol{\eta}^{\ell}(\boldsymbol{x},t)$ is defined in terms of its Lagrangian coordinate $\boldsymbol{x} = \boldsymbol{a}$, it is often more convenient to adopt an Eulerian description $\boldsymbol{\eta}(\boldsymbol{x}^{\ell},t)$ of the displacement in terms of the Eulerian coordinate $\boldsymbol{x}^{\ell}(\boldsymbol{x},t)$ from which the particle suffers its infinitesimal displacement. Consistent with our superscript ℓ notation (2.4), we define $\boldsymbol{\eta}$ implicitly by

$$\boldsymbol{\eta}^{\ell}(\boldsymbol{x},t) = \boldsymbol{\eta}(\boldsymbol{x}^{\ell}(\boldsymbol{x},t),t)\,. \tag{2.12}$$

We write $\delta\phi$ for the Eulerian increment of a function $\phi(\boldsymbol{x},t)$ at fixed $\boldsymbol{x}$ caused by the virtual displacement field $\boldsymbol{\eta}(\boldsymbol{x},t)$. It follows from this definition that the Lagrangian increment $\delta(\phi^{\ell})$ of the function $\phi(\boldsymbol{x}^{\ell}(\boldsymbol{x},t),t)$ following a trajectory from a fixed initial position $\boldsymbol{x}=\boldsymbol{a}$ is

$$\delta(\phi^{\ell}) = (\delta\phi + \boldsymbol{\eta}\cdot\nabla\phi)^{\ell}\,. \tag{2.13a}$$

Since our virtual displacement does not preserve volume, the increment of $\mathscr{J}$ is given by

$$\frac{\delta\mathscr{J}}{\mathscr{J}} = \frac{\partial x_k}{\partial x_j^{\ell}}\frac{\partial \eta_j^{\ell}}{\partial x_k} \equiv (\nabla\cdot\boldsymbol{\eta})^{\ell}\,, \tag{2.13b}$$

where we have involved a 'chain rule' similar to (2.5a). The result enables us to express (2.13a) in the alternative form

$$\delta(\mathscr{J}\phi^{\ell}) = \mathscr{J}(\delta\phi + \nabla\cdot(\boldsymbol{\eta}\phi))^{\ell}. \tag{2.13c}$$

Since mass is conserved under the virtual displacement, it follows from (2.8b) that $\delta(\mathscr{J}\rho^{\ell}) = 0$. In view of our intended interpretation of the Lagrange multiplier p as the Eulerian pressure field, we will assume that this field is unaltered under the virtual displacement, i.e. $\delta p = 0$. To obtain the increments of ρ, p and $\boldsymbol{v}$ following a trajectory, use of (2.13a,b) leads to the values

$$\delta(\rho^{\ell}) = -(\rho\nabla\cdot\boldsymbol{\eta})^{\ell}\,, \qquad \delta(p^{\ell}) = (\boldsymbol{\eta}\cdot\nabla\, p)^{\ell}, \tag{2.14a,b}$$

while from (2.6a,b) we deduce that

$$\delta(\boldsymbol{v}^{\ell}) = \partial_t(\boldsymbol{\eta}^{\ell}) = (\mathscr{D}_t\,\boldsymbol{\eta})^{\ell}\,. \tag{2.14c}$$

The implementation of Hamilton's principle requires that we determine the action increment

$$\delta\mathscr{A} = \int_{t_0}^{t_1}\delta\mathscr{L}\,\mathrm{d}t\,, \qquad \text{where} \qquad \delta\mathscr{L} = \int_{\mathscr{V}_0}\delta(\mathscr{J}L^{\ell})\,\mathrm{d}^3x \tag{2.15a,b}$$

(see (2.10)). Since (2.13c) implies

$$\mathscr{J}(\delta L)^{\ell} = \delta(\mathscr{J}L^{\ell}) - \mathscr{J}(\nabla\cdot(L\boldsymbol{\eta}))^{\ell}, \tag{2.15c}$$

we may also write (2.15b) in the more convenient form

$$\delta\mathscr{L} = \int_{\mathscr{V}_0} \mathscr{J}(\delta L)^\ell \, \mathrm{d}^3x = \int_{\mathscr{V}^\ell} (\delta L)^\ell \, \mathrm{d}^3x^\ell , \qquad (2.15d)$$

because the contribution from $\int_{\mathscr{V}^\ell}(\nabla\cdot(L\boldsymbol{\eta}))^\ell \mathrm{d}x^\ell$ vanishes on application of the divergence theorem and boundary conditions on $\boldsymbol{\eta}^\ell$. We evaluate $\mathscr{J}(\delta L)^\ell$ by substituting the values

$$\mathscr{J}L^\ell = \tfrac{1}{2}\rho_0 \left|\boldsymbol{v}^\ell\right|^2 - p^\ell(1-\mathscr{J}) \qquad \text{and} \qquad L = p - \rho\,\Pi , \qquad (2.16a, b)$$

determined by (2.8b) and (2.9), into the right-hand side of (2.15c). The result is

$$\begin{aligned}\mathscr{J}(\delta L)^\ell &= \rho_0\boldsymbol{v}^\ell\cdot\delta(\boldsymbol{v}^\ell) + p^\ell\delta\mathscr{J} - (1-\mathscr{J})\,\delta(p^\ell) - \mathscr{J}(\nabla\cdot((p - \rho\,\Pi)\boldsymbol{\eta}))^\ell \\ &= -\rho_0\,\boldsymbol{\eta}^\ell\cdot\boldsymbol{\mathscr{E}}^\ell + \partial_t(\rho_0\boldsymbol{\eta}^\ell\cdot\boldsymbol{v}^\ell) + \mathscr{J}(\nabla\cdot(\rho\boldsymbol{\eta}\Pi))^\ell , \end{aligned} \qquad (2.16c)$$

in which we have used (2.13b), (2.14) and where $\boldsymbol{\mathscr{E}}$ is defined by (2.3b). For virtual trajectories both the time integral $\int_{t_0}^{t_1}\partial_t\big(\rho_0\boldsymbol{\eta}^\ell\cdot\boldsymbol{v}^\ell\big)\mathrm{d}t$ and the volume integral $\int_{\mathscr{V}^\ell}(\nabla\cdot(\rho\boldsymbol{\eta}\Pi))^\ell \mathrm{d}^3x^\ell$ vanish because of the temporal end-point and boundary conditions. That leaves

$$\delta\mathscr{A} = -\int_{t_0}^{t_1}\int_{\mathscr{V}_0} \rho_0\boldsymbol{\eta}^\ell\cdot\boldsymbol{\mathscr{E}}^\ell \, \mathrm{d}^3x \;\mathrm{d}t . \qquad (2.17)$$

Hamilton's principle requires that $\delta\mathscr{A} = 0$ for all such $\boldsymbol{\eta}^\ell$ and Euler's equation $\boldsymbol{\mathscr{E}} = \boldsymbol{0}$, namely (2.3$a$), follows (albeit evaluated at the particle position $\boldsymbol{x}^\ell$ rather than $\boldsymbol{x}$). Furthermore, the variational calculation now being complete, we can set $\rho^\ell = \rho_0$ so that mass continuity reduces to the statement (2.2).

These ideas are developed further in the hybrid Eulerian–Lagrangian approach of §3.1 below.

2.2. *The Eulerian description*

Since Hamilton's principle builds on the notion of particle paths, the 'first principles' derivation of Euler's equation in the previous subsection was naturally Lagrangian in nature, relying on the labelling of particle paths by their initial position $\boldsymbol{a}$. It is not, however, necessary to remained locked to that point of view. Essentially all that we need is the value of the increment of the first integral $\mathscr{L} = \int_{\mathscr{V}^\ell} L^\ell \,\mathrm{d}^3x^\ell$ in (2.10b) subject to the particle path constraint. So instead of considering Lagrangian increments $\delta(\phi^\ell)$ at points $\boldsymbol{x}^\ell(\boldsymbol{a}, t)$ following the trajectory of a fluid particle, we consider the Eulerian increments $(\delta\phi)^\ell$ at fixed position $\boldsymbol{x}^\ell$. So from (2.13a) and (2.14a, c), the Eulerian velocity and density increments at $\boldsymbol{x}^\ell$ are

$$(\delta\boldsymbol{v})^\ell = (\mathscr{D}_t\,\boldsymbol{\eta} - \boldsymbol{\eta}\cdot\nabla\boldsymbol{v})^\ell , \qquad (\delta\rho)^\ell = -(\nabla\cdot(\rho\boldsymbol{\eta}))^\ell . \qquad (2.18a, b)$$

According to our variational assumption the Eulerian pressure increment vanishes ($(\delta p)^\ell = 0$) and so the corresponding increment of the Lagrangian (2.9b) is

$$(\delta L)^\ell = -\rho^\ell(\delta\Pi)^\ell - \Pi^\ell(\delta\rho)^\ell , \qquad \text{where} \qquad (\delta\Pi)^\ell = -\boldsymbol{v}^\ell\cdot(\delta\boldsymbol{v})^\ell \qquad (2.19a, b)$$

on use of (2.9c).

Since it is notationally inconvenient to use $\boldsymbol{x}^\ell$ as the Eulerian coordinate, we replace it by $\boldsymbol{x}$. In consequence, functions of the form ϕ^ℓ and $(\delta\phi)^\ell$ become simply ϕ and $\delta\phi$. For obvious reasons, we call this change of notation 'dropping the ℓ'. In application of the idea it is important to realize that the ℓ may only be dropped when it is the final operation applied. For example, we can drop the ℓ from $(\delta\phi)^\ell$ but cannot drop it from $\delta(\phi^\ell)$ because that δ–operation is applied after the ℓ–operation. On dropping the ℓ in the integrand of $\int_{\mathscr{V}^\ell}(\delta L)^\ell\,\mathrm{d}^3x^\ell$ (see (2.15*d*)), we obtain the increment $\delta\mathscr{A}[\boldsymbol{v},\ \rho,\ p]$ of the action $\mathscr{A}[\boldsymbol{v},\ \rho,\ p]$ as

$$\delta\mathscr{A}[\boldsymbol{v},\ \rho,\ p] = \int_{t_0}^{t_1} \delta\mathscr{L}\,\mathrm{d}t\,, \qquad \text{where} \qquad \delta\mathscr{L} = \int_{\mathscr{V}^\ell} \delta L\,\mathrm{d}^3x\,; \tag{2.20a,b}$$

the arguments $\boldsymbol{v}$, ρ, and p are listed explicitly to emphasize the functions upon which the action increment now depends. Some care needs to be taken with regard to the interpretation of the volume integral, in which the ℓ has been dropped in the integrand but not on the name $\mathscr{V}^\ell$ of the region involved as it not stationary but moves with the fluid. From (2.19) the increment of the Lagrangian density $\mathscr{L}$ is

$$\delta L = \rho\boldsymbol{v}\cdot\delta\boldsymbol{v} - \Pi\,\delta\rho\,. \tag{2.20c}$$

On dropping the ℓ in (2.18), we obtain

$$\delta\boldsymbol{v} = \partial_t\boldsymbol{\eta} + [\boldsymbol{v}\,,\,\boldsymbol{\eta}]\,, \qquad \delta\rho = -\nabla\cdot(\rho\boldsymbol{\eta})\,, \tag{2.21a,b}$$

where the notation $[\,\cdot\,,\,\cdot\,]$ means

$$[\boldsymbol{v}\,,\,\boldsymbol{\eta}] \equiv \boldsymbol{v}\cdot\nabla\boldsymbol{\eta} - \boldsymbol{\eta}\cdot\nabla\boldsymbol{v}\,. \tag{2.21c}$$

On forming the scalar product of $\delta\boldsymbol{v}$ with an arbitrary vector field $\boldsymbol{V}(\boldsymbol{x},t)$, (2.21*a*) yields the useful relation

$$\boldsymbol{V}\cdot\delta\boldsymbol{v} = \boldsymbol{V}\cdot(\partial_t\boldsymbol{\eta} + [\boldsymbol{v}\,,\,\boldsymbol{\eta}]) = -\boldsymbol{\eta}\cdot(\partial_t\boldsymbol{V} + \{\boldsymbol{v}\,,\,\boldsymbol{V}\}) + \mathscr{D}_t(\boldsymbol{\eta}\cdot\boldsymbol{V})\,, \tag{2.22a}$$

in which the notation $\{\,\cdot\,,\,\cdot\,\}$ means

$$\{\boldsymbol{v}\,,\,\boldsymbol{V}\} \equiv \boldsymbol{v}\cdot\nabla\,\boldsymbol{V} + (\nabla\,\boldsymbol{v})\cdot\boldsymbol{V}\,, \tag{2.22b}$$

where in component form $((\nabla\,\boldsymbol{v})\cdot\boldsymbol{V})_i = (\partial v_k/\partial x_i)V_k$ (see Appendix A, (A2*a*)). Though our immediate applications in this section involve the choice $\boldsymbol{V} = \boldsymbol{v}$, we introduce this more general notation for later use in §3.

Substitution of the values of $\delta\boldsymbol{v}$ and $\delta\rho$ given by (2.21*a*, *b*) into (2.20*c*) and use of (2.22*a*) with $\boldsymbol{V} = \boldsymbol{v}$ determines the increment

$$\delta L = -\boldsymbol{\eta}\cdot\rho\boldsymbol{\mathscr{E}} + \rho\mathscr{D}_t\big(\boldsymbol{\eta}\cdot\boldsymbol{v}\big) + \nabla\cdot(\boldsymbol{\eta}\rho\Pi)\,, \tag{2.23a}$$

where $\boldsymbol{\mathscr{E}}$ emerges naturally in the form

$$\boldsymbol{\mathscr{E}} \equiv \partial_t\boldsymbol{v} + \{\boldsymbol{v}\,,\,\boldsymbol{v}\} + \nabla\Pi\,. \tag{2.23b}$$

As in the Lagrangian case of §2.1 above, it is important to appreciate that, though the fluid is incompressible, the virtual displacement is allowed to cause density variations ($\delta\rho \neq 0$). The only constraint on ρ, until all the variations have been completed, is that it obeys the mass continuity equation

$$\partial_t\rho + \nabla\cdot(\rho\boldsymbol{v}) = 0\,. \tag{2.24}$$

In preparation for our analysis in §3.2 below, we note that use of this equation allows us to rewrite (2.23*a*) as

$$\delta L = -\boldsymbol{\eta}\cdot\rho\boldsymbol{\mathscr{E}} + \partial_t(\boldsymbol{\eta}\cdot\rho\boldsymbol{v}) + \nabla\cdot((\boldsymbol{\eta}\cdot\rho\boldsymbol{v})\,\boldsymbol{v} + \boldsymbol{\eta}\rho\Pi)\,, \tag{2.25a}$$

with (2.23*b*) expressed in the alternative form

$$\rho\boldsymbol{\mathscr{E}} = \partial_t(\rho\boldsymbol{v}) + \langle\boldsymbol{v}\,,\,\rho\boldsymbol{v}\rangle + \rho\,\nabla\Pi\,, \tag{2.25b}$$

in which the notation $\langle\,\cdot\,,\,\cdot\,\rangle$ is defined in Appendix A by (A 2*b*). On returning to the more primitive form (2.23*b*) we see that, since $\mathscr{J}\rho^{\ell} = \rho_0$ (see (2.8*b*)), the value of $\mathscr{J}(\delta L)^{\ell}$ determined by (2.23*a*) coincides with our earlier result (2.16*c*). Accordingly the action increment $\delta\mathscr{A}$ determined by (2.20) in our Eulerian analysis also agrees with that, (2.17), obtained by following particle paths. This means that the vanishing of $\delta\mathscr{A}$ obtained from our Eulerian point of view recovers $\boldsymbol{\mathscr{E}} = \boldsymbol{0}$.

Significantly, in relation to our developments in the next section, we may write expression (2.23*b*) for $\boldsymbol{\mathscr{E}}$ as

$$\boldsymbol{\mathscr{E}} = \frac{\partial}{\partial t}\left(\frac{1}{\rho}\frac{\delta\mathscr{A}}{\delta\boldsymbol{v}}\right) + \left\{\boldsymbol{v}\,,\,\left(\frac{1}{\rho}\frac{\delta\mathscr{A}}{\delta\boldsymbol{v}}\right)\right\} - \nabla\left(\frac{\delta\mathscr{A}}{\delta\rho}\right) \tag{2.26a}$$

in terms of the variational derivatives

$$\frac{1}{\rho}\frac{\delta\mathscr{A}}{\delta\boldsymbol{v}} = \boldsymbol{v}\,, \qquad -\frac{\delta\mathscr{A}}{\delta\rho} = \Pi \tag{2.26b, c}$$

of the functional $\mathscr{A}[\boldsymbol{v},\ \rho,\ p]$ (see the action increment (2.20*a*)) obtained trivially from (2.20*c*). Recalling the explicit form (2.9*a*) of the Lagrangian, we note that the p variation of the action gives

$$\frac{\delta\mathscr{A}}{\delta p} = \frac{\rho}{\rho_0} - 1\,. \tag{2.27}$$

Its vanishing recovers the fact that the fluid has constant density $\rho = \rho_0$ with the consequence that (2.24) reduces to (2.2).

Finally, when the virtual displacements are material line elements $\boldsymbol{\eta} = \mathrm{d}\boldsymbol{x}$ and the flow is unperturbed $\delta\boldsymbol{v} = \boldsymbol{0}$, (2.22*a*) shows that the material derivative of the scalar product $\boldsymbol{V}\cdot\mathrm{d}\boldsymbol{x}$ (for arbitrary $\boldsymbol{V}$) is

$$\mathscr{D}_t\,(\boldsymbol{V}\cdot\mathrm{d}\boldsymbol{x}) = \big(\partial_t\boldsymbol{V} + \{\boldsymbol{v}\,,\ \boldsymbol{V}\}\big)\cdot\mathrm{d}\boldsymbol{x}\,. \tag{2.28}$$

This result explains some of the attraction of Euler's equation $\boldsymbol{\mathscr{E}} = \boldsymbol{0}$ with the representation (2.23*b*) of $\boldsymbol{\mathscr{E}}$ for, when $\boldsymbol{V} = \boldsymbol{v}$, together they yield yet another form

$$\mathscr{D}_t\,(\boldsymbol{v}\cdot\mathrm{d}\boldsymbol{x}) = -\,\mathrm{d}\,\Pi \tag{2.29a}$$

of Euler's equation. It may used to derive Kelvin's circulation theorem

$$\frac{\mathrm{d}}{\mathrm{d}t}\oint_{\mathscr{C}^{\ell}}\boldsymbol{v}\cdot\mathrm{d}\boldsymbol{x} = 0 \tag{2.29b}$$

for material closed curves $\mathscr{C}^{\ell}$ composed of points $\boldsymbol{x}$, that move move with velocity $\boldsymbol{v}(\boldsymbol{x}, t)$.

3. Flow decompositions

The hybrid Eulerian–Lagrangian approach, pioneered by Soward (1972) in the MHD context, and inspired by Braginsky's (1964) development of kinematic dynamo theory, provides the motivation for the flow decompositions that we discuss in this section. Our survey in §3.1 closely follows Andrews & McIntyre (1978*a*) and also Holm's (2002*b*) subsequent development that stresses its variational basis. Only after averages are taken, as outlined in the next subsection, do we reach the generalized Lagrangian–mean description of Andrews & McIntyre (1978*a*) that Holm (2002*b*) later refers to as the generalized lagrangian mean (GLM) approach. The complication that plagues the GLM approach is that all variables refer to properties at displaced points (the hybrid Eulerian–Lagrangian (HEL) description). As we explain in §3.2, Holm (2002*b*) attempts to bypass that difficulty by taking what is best of the HEL approach by using variables that refer to properties at the actual undisplaced points (the Eulerian description). Of course, that is exactly what Braginsky (1964) and later Tough & Roberts (1968) were aiming for in their use of so-called 'effective variables'.

Subsection 3.2 is the cornerstone of our paper. There we follow Holm (2002*b*) by starting from Hamilton's principle. We show however that Holm's variational procedures are in error because they fail to adhere to Hamilton's recipe correctly. As a result, key terms in Euler's equation are lost. This is a significant result because the equation, which Holm obtains provides the basis of his so-called 'glm-approach'. We discuss the consequences of this later.

A significant aspect of our new development concerns the size of the displacements $\boldsymbol{\xi}$ and $\boldsymbol{\zeta}$, which we introduce in §3.1 and 3.2. We therefore stress that, throughout this section and the following §4, no assumption is made about the magnitude of these displacement vectors and so the results obtained are completely general. Only in §5 do we follow Holm in making approximations based on small-amplitude displacements in order to make careful comparisons with his results.

3.1. *A hybrid Eulerian–Lagrangian (HEL) representation*

Though the Lagrangian description of the flow introduced in §2.1 has many attractions, not least that it provides the natural framework to apply Hamilton's principle, it is generally unwieldy to work with because $\boldsymbol{x}^\ell(\boldsymbol{a},t)$ moves ever further away from its starting point $\boldsymbol{a}$, while the deformation matrix $\partial x_i^\ell/\partial x_j(\boldsymbol{a},t)$ generally increases in concert. This means that problem-solving based on Euler's equation $\boldsymbol{\mathscr{E}}^\ell=\boldsymbol{0}$ for the functions $\boldsymbol{v}^\ell(\boldsymbol{x},t)$ and $\boldsymbol{x}^\ell(\boldsymbol{x},t)$ is not feasible. Even the kinematic task of determining closed-form solutions for the particle paths $\boldsymbol{x}^\ell(\boldsymbol{x},t)$ from an initial position $\boldsymbol{x}=\boldsymbol{a}$ with given $\boldsymbol{v}^\ell(\boldsymbol{x},t)$ is generally intractable.

To avoid the difficulties mentioned, we will develop the HEL approach; it works in the following way. We introduce a fictitious (or reference) flow, for which the position $\boldsymbol{x}^\Lambda(\boldsymbol{a},t)$ reached by a reference fluid particle is distinct from the true particle position $\boldsymbol{x}^\ell(\boldsymbol{a},t)$. So instead of considering a map $\boldsymbol{a}\mapsto\boldsymbol{x}^\ell$ as in §2, we partition it into two and consider the map $\boldsymbol{a}\mapsto\boldsymbol{x}^\Lambda$ followed by $\boldsymbol{x}^\Lambda\mapsto\boldsymbol{x}^\ell$. Our objective is to express the equation of motion at the Eulerian coordinate $\boldsymbol{x}^\ell$ in terms of the HEL coordinate $\boldsymbol{x}^\Lambda$ rather than the Lagrangian coordinate $\boldsymbol{a}$.

For the first map $\boldsymbol{a}\mapsto\boldsymbol{x}^\Lambda$, we note that the kinematic apparatus set up in §2.1 for the real flow can be taken over largely intact for the reference flow. So we define the Λ–operation in the same way, use $\boldsymbol{x}$ for the Lagrangian coordinate and write

$$\phi^\Lambda(\boldsymbol{x},t)=\phi(\boldsymbol{x}^\Lambda(\boldsymbol{x},t),t) \tag{3.1}$$

(cf. (2.4)) while the gradient ∇^Λ is defined by (2.5) with $\boldsymbol{x}^\ell$ replaced throughout by $\boldsymbol{x}^\Lambda$. We introduce $\boldsymbol{u}^\Lambda(\boldsymbol{x},t)$ to describe the motion of the reference flow at $\boldsymbol{x}^\Lambda(\boldsymbol{x},t)$, which, like $\boldsymbol{v}^\ell(\boldsymbol{x},t)$ (see (2.6*b*)), is defined by

$$\boldsymbol{u}^\Lambda(\boldsymbol{x},t)\equiv\boldsymbol{u}(\boldsymbol{x}^\Lambda(\boldsymbol{x},t),t)=\partial_t(\boldsymbol{x}^\Lambda)\,. \tag{3.2}$$

This provides an implicit definition of the velocity $\boldsymbol{u}(\boldsymbol{x},t)$, that is central to the analysis that follows. In particular, by analogy with (2.6*a*), the time derivative of a function $\phi^\Lambda(\boldsymbol{x},t)=\phi(\boldsymbol{x}^\Lambda(\boldsymbol{x},t),t)$ is

$$\partial_t(\phi^\Lambda)=(\mathrm{D}_t\phi)^\Lambda\,,\qquad \text{where}\qquad \mathrm{D}_t\equiv\partial_t+\boldsymbol{u}\boldsymbol{\cdot}\nabla \tag{3.3a,b}$$

is the material derivative, replacing (2.1*b*), associated with the reference flow velocity $\boldsymbol{u}$. In order to be clear on the status of the reference velocity field $\boldsymbol{u}(\boldsymbol{x},t)$, we note that it coincides with the real velocity field $\boldsymbol{v}(\boldsymbol{x},t)$ in the case of the pure Lagrangian description, when $\boldsymbol{x}^{\Lambda}(\boldsymbol{x},t)=\boldsymbol{x}^{\ell}(\boldsymbol{x},t)$.

To accomplish the second map $\boldsymbol{x}^{\Lambda}\mapsto\boldsymbol{x}^{\ell}$ we introduce the coordinate shift function $\boldsymbol{x}^{L}(\boldsymbol{x},t)$ with the property that the true particle position $\boldsymbol{x}^{\ell}$ is reached after application of the Λ–operation:

$$\boldsymbol{x}^{\ell}(\boldsymbol{x},t) = (\boldsymbol{x}^{L})^{\Lambda}(\boldsymbol{x},t) = \boldsymbol{x}^{L}(\boldsymbol{x}^{\Lambda}(\boldsymbol{x},t),t). \tag{3.4a}$$

In what follows, $\boldsymbol{x}^{L}(\boldsymbol{x},t)$ will be written

$$\boldsymbol{x}^{L}(\boldsymbol{x},t) = \boldsymbol{x} + \boldsymbol{\xi}(\boldsymbol{x},t), \tag{3.4b}$$

the advantage of which becomes apparent in §5 where $|\boldsymbol{\xi}|$ is assumed to be small. We emphasize, however, that the development in this subsection does not depend on that smallness and is therefore completely general.

We call functions

$$\phi^{L}(\boldsymbol{x},t) \equiv \phi(\boldsymbol{x}^{L}(\boldsymbol{x},t),t), \tag{3.5a}$$

evaluated at $\boldsymbol{x}^{L}$, HEL variables, while the position $\boldsymbol{x}$, upon which $\boldsymbol{x}^{L}$ depends, is the HEL coordinate. They have the property that the value of ϕ at the true particle position $\boldsymbol{x}^{\ell}$ is obtained on application of the Λ–operation:

$$\phi^{\ell}(\boldsymbol{x},t)=(\phi^{L})^{\Lambda}(\boldsymbol{x},t)=\phi^{L}(\boldsymbol{x}^{\Lambda}(\boldsymbol{x},t),t)= \phi((\boldsymbol{x}^{L})^{\Lambda}(\boldsymbol{x},t),t)=\phi(\boldsymbol{x}^{\ell}(\boldsymbol{x},t),t). \tag{3.5b}$$

From this point of view, the Lagrangian variable ϕ^{ℓ} and the HEL variable ϕ^{L} are identical, provided the argument of the former is the Lagrangian coordinate $\boldsymbol{x}=\boldsymbol{a}$ and the argument of the latter is the HEL coordinate $\boldsymbol{x}=\boldsymbol{x}^{\Lambda}(\boldsymbol{a},t)$; that is the sense in which we use the notation. To effect the change of dependent variable from $\boldsymbol{a}$ to $\boldsymbol{x}^{\Lambda}$, we introduce the 'dropping the Λ' process by analogy with the 'dropping the ℓ' process described below (2.19).

We begin the application of our apparatus to the Eulerian velocity field $\boldsymbol{v}(\boldsymbol{x},t)=\mathscr{D}_t\boldsymbol{x}$ (see (2.1*b*)). The corresponding Lagrangian velocity $\boldsymbol{v}^{\ell}(\boldsymbol{x},t)=\partial_t(\boldsymbol{x}^{\ell})$ is obtained by setting $\boldsymbol{x}^{\ell}=(\boldsymbol{x}^{L})^{\Lambda}$ and replacing ϕ in (3.3*a*) by $\boldsymbol{x}^{L}$. The result is

$$\boldsymbol{v}^{\ell} = \partial_t\left((\boldsymbol{x}^{L})^{\Lambda}\right) = (\mathrm{D}_t\boldsymbol{x}^{L})^{\Lambda}, \tag{3.6a}$$

which, since $\boldsymbol{v}^{\ell}=(\boldsymbol{v}^{L})^{\Lambda}$, establishes the equality $(\boldsymbol{v}^{L})^{\Lambda}=(\mathrm{D}_t\boldsymbol{x}^{L})^{\Lambda}$. Since the Λ–operation is the last to be applied on both sides of the equality, the Λ may be dropped yielding the HEL velocity

$$\boldsymbol{v}^{L} = (\mathscr{D}_t\boldsymbol{x})^{L} = \mathrm{D}_t\boldsymbol{x}^{L} = \boldsymbol{u} + \mathrm{D}_t\boldsymbol{\xi}. \tag{3.6b}$$

Likewise the Eulerian form $\mathscr{D}_t\phi$ for the material derivative of ϕ, when evaluated at $\boldsymbol{x}^{\ell}$, is $(\mathscr{D}_t\phi)^{\ell}(\boldsymbol{x},t)=\partial_t(\phi^{\ell})$ (see (2.6*a*)). So on writing $\phi^{\ell}=(\phi^{L})^{\Lambda}$ and replacing ϕ in (3.3*a*) by ϕ^{L}, we obtain the alternative identity $\partial_t(\phi^{\ell})=(\mathrm{D}_t\phi^{L})^{\Lambda}$. Then recalling that $(\mathscr{D}_t\phi)^{\ell}=((\mathscr{D}_t\phi)^{L})^{\Lambda}$, the two separate representations for $\partial_t(\phi^{\ell})$ determine the identity $((\mathscr{D}_t\phi)^{L})^{\Lambda}=(\mathrm{D}_t\phi^{L})^{\Lambda}$. Finally, dropping the Λ determines the HEL material derivative

$$(\mathscr{D}_t\phi)^{L} = \mathrm{D}_t\phi^{L}. \tag{3.7}$$

Linked to the reference flow $\boldsymbol{u}$, defined by (3.2), we introduce the concept of the reference mass density σ. Just as we define $\boldsymbol{u}^{\Lambda}$ in (3.2), we write $\sigma^{\Lambda}(\boldsymbol{x},t)=\sigma(\boldsymbol{x}^{\Lambda}(\boldsymbol{x},t),t)$ to identify the value of σ at $\boldsymbol{x}^{\Lambda}(\boldsymbol{x},t)$. Accordingly reference mass conservation requires that the mass element $\left(\sigma^{\Lambda}\mathrm{d}^3x^{\Lambda}\right)(\boldsymbol{x},t)$ advected by the reference flow from the initial position $\boldsymbol{x}=\boldsymbol{a}$ at time $t=t_0$ remains unaltered and continues to take the initial value $\sigma(\boldsymbol{x},t_0)\,\mathrm{d}^3x$. This corresponds to the result (2.8*a*), namely $\left(\rho^{\ell}\mathrm{d}^3x^{\ell}\right)(\boldsymbol{x},t)=\rho_0\,\mathrm{d}^3x$, for the real flow. Though the choice of initial reference density $\sigma(\boldsymbol{x},t_0)$ is not unique, it is natural to assume that it coincides with the initial real fluid density ρ_0. So, since both our conserved masses $\rho^{\ell}\mathrm{d}^3x^{\ell}=(\rho^L\mathrm{d}^3x^L)^{\Lambda}$ and $\sigma^{\Lambda}\mathrm{d}^3x^{\Lambda}=(\sigma\,\mathrm{d}^3x)^{\Lambda}$ are equal to $\rho_0\,\mathrm{d}^3x$, we have $(\sigma\,\mathrm{d}^3x)^{\Lambda}=(\rho^L\mathrm{d}^3x^L)^{\Lambda}$, which on dropping the Λ gives

$$\sigma\,\mathrm{d}^3x=\rho^L\mathrm{d}^3x^L \qquad \text{implying} \qquad J\rho^L=\sigma\,, \tag{3.8a,b}$$

where $J(\boldsymbol{x},t)=\det(\partial x_i^L/\partial x_j)$ is the Jacobian and $\boldsymbol{x}$ is now the HEL rather than the Lagrangian coordinate. We remark that, even for our incompressible case, it is generally convenient to allow for dilatation $J\neq 1$ of the reference flow. Accordingly, σ may vary and consequently $\boldsymbol{u}$ is not solenoidal, $\nabla\boldsymbol{\cdot}\boldsymbol{u}\neq 0$. So, unlike the true Lagrangian case $\boldsymbol{u}=\boldsymbol{0}$, the density σ is not simply ρ_0 but evolves with time, satisfying the mass continuity equation

$$\partial_t\sigma+\nabla\boldsymbol{\cdot}(\sigma\boldsymbol{u})=0\,. \tag{3.9}$$

At this point, it is important to appreciate that the flow description in the HEL approach is not unique, a feature that is transparent from the particle-path description $\boldsymbol{x}^{\ell}(\boldsymbol{a},t)=\boldsymbol{x}^L(\boldsymbol{x}^{\Lambda}(\boldsymbol{a},t),t)$ involving two independent functions $\boldsymbol{x}^L=\boldsymbol{x}^L(\boldsymbol{x},t)$ and $\boldsymbol{x}^{\Lambda}=\boldsymbol{x}^{\Lambda}(\boldsymbol{x},t)$. So the specification of the virtual displacement $\boldsymbol{\eta}^{\ell}(\boldsymbol{a},t)=\boldsymbol{\eta}(\boldsymbol{x}^L(\boldsymbol{x}^{\Lambda}(\boldsymbol{a},t),t)$ of the particle path is not unique either; we may vary either $\boldsymbol{x}^L$ or $\boldsymbol{x}^{\Lambda}$ alone, or both simultaneously. To appreciate the structure, we consider the general case of simultaneous variation, for which $\boldsymbol{x}^{\Lambda}$ and $\boldsymbol{x}^L$ are subject to the infinitesimal increments $\boldsymbol{\mu}^{\Lambda}$ and $\delta(\boldsymbol{x}^L)$ respectively.

Linked to the reference flow $\boldsymbol{u}$, defined by (3.2), we may introduce the concept of its associated virtual displacement field $\boldsymbol{\mu}$ such that the reference particle displacement at $\boldsymbol{x}^{\Lambda}$ as a function of the Lagrangian coordinate $\boldsymbol{x}$ is

$$\boldsymbol{\mu}^{\Lambda}(\boldsymbol{x},t)=\boldsymbol{\mu}(\boldsymbol{x}^{\Lambda}(\boldsymbol{x},t),t) \tag{3.10}$$

similar to the definition (2.12) for the real particle displacement $\boldsymbol{\eta}$. Then increments of functions $\phi^{\Lambda}(\boldsymbol{x},t)$ at fixed $\boldsymbol{x}$ due to virtual reference particle displacements $\boldsymbol{\mu}$ are given exactly as in (2.13*a*) with the $\boldsymbol{\eta}$ replaced by $\boldsymbol{\mu}$:

$$\delta(\phi^{\Lambda})=(\delta\phi+\boldsymbol{\mu}\boldsymbol{\cdot}\nabla\phi)^{\Lambda}\,. \tag{3.11}$$

Then repeating the arguments, that lead to the formulae (2.21*a*, *b*) for the increments of $\boldsymbol{v}$ and ρ of the real flow, gives

$$\delta\boldsymbol{u}=\partial_t\boldsymbol{\mu}+[\boldsymbol{u}\,,\,\boldsymbol{\mu}]\,, \qquad \delta\sigma=-\nabla\boldsymbol{\cdot}(\sigma\boldsymbol{\mu}) \tag{3.12a,b}$$

for the increments of the reference velocity and density, where, having dropped the Λ, $\boldsymbol{x}$ is now the HEL coordinate.

The virtual displacement for the real flow may be expressed as $\boldsymbol{\eta}^{\ell}=\delta(\boldsymbol{x}^{\ell})$ or equivalently as $(\boldsymbol{\eta}^L)^{\Lambda}=\delta((\boldsymbol{x}^L)^{\Lambda})$. Then application of the formula (3.11) to the increment followed by dropping the Λ leads, on use of (3.4*b*), to the HEL representation of the virtual path displacement

$$\boldsymbol{\eta}^L=\delta\boldsymbol{\xi}+\boldsymbol{\mu}\boldsymbol{\cdot}\nabla\boldsymbol{x}^L\,, \qquad \text{in which} \qquad \delta\boldsymbol{\xi}=\delta(\boldsymbol{x}^L)\,. \tag{3.13}$$

Likewise the Lagrangian velocity increment $\delta(\boldsymbol{v}^{\ell})=\partial_t(\boldsymbol{\eta}^{\ell})$ may be written, with the help of (3.3*a*), in the form $\delta((\boldsymbol{v}^L)^{\Lambda})=\partial_t((\boldsymbol{\eta}^L)^{\Lambda})=(\mathrm{D}_t\boldsymbol{\eta}^L)^{\Lambda}$. Evaluation of $\delta((\boldsymbol{v}^L)^{\Lambda})$

using (3.11) leads, after the Λ is dropped, to the Lagrangian velocity increment

$$\delta(\boldsymbol{v}^L) + \boldsymbol{\mu}\boldsymbol{\cdot}\nabla\boldsymbol{v}^L = \mathrm{D}_t(\delta\boldsymbol{\xi} + \boldsymbol{\mu}\boldsymbol{\cdot}\nabla\boldsymbol{x}^L) \tag{3.14a}$$

resulting from following the material particle displacement. From this we deduce that the HEL velocity increment, at fixed HEL coordinate $\boldsymbol{x}$, may be expressed in terms of the increments $\delta\boldsymbol{\xi}$ and $\delta\boldsymbol{u}$ alone as

$$\delta(\boldsymbol{v}^L) = \mathrm{D}_t\,\delta\boldsymbol{\xi} + \delta\boldsymbol{u}\boldsymbol{\cdot}\nabla\boldsymbol{x}^L\,. \tag{3.14b}$$

Though this result can also be obtained directly from incrementing $\boldsymbol{v}^L$ as defined by (3.6*b*), the above longer derivation has enabled us to develop the general structure that we need to apply to Hamilton's principle.

The original incrementation (2.13*a*) in our new notation is

$$\delta(\phi^\ell) = ((\delta\phi)^L)^\Lambda + ((\boldsymbol{\eta}\boldsymbol{\cdot}\nabla\phi)^L)^\Lambda\,. \tag{3.15a}$$

An alternative expression of $\delta(\phi^\ell) = \delta((\phi^L)^\Lambda)$ is obtained from (3.11) which, together with (3.13) and after dropping the Λ, determines the HEL increment

$$\delta(\phi^L) = (\delta\phi)^L + \delta\boldsymbol{\xi}\boldsymbol{\cdot}\nabla^L\phi^L\,. \tag{3.15b}$$

So, since the Eulerian pressure increment vanishes $(\delta p)^L = 0$, the HEL pressure increment is

$$\delta(p^L) = \delta\boldsymbol{\xi}\boldsymbol{\cdot}\nabla^L p^L \qquad \text{and} \qquad \delta J = J\,\nabla^L\boldsymbol{\cdot}\,\delta\boldsymbol{\xi}\,, \tag{3.16a, b}$$

similar to (2.14*b*) and (2.13*b*); recall that we are using the L–operation in the sense of the definition (3.5*a*).

In our new notation, the action increment (2.15*d*) becomes

$$\delta\mathscr{L} = \int_{\mathscr{V}^\ell} (\delta L)^\ell\,\mathrm{d}^3x^\ell = \int_{\mathscr{V}^\Lambda} \left((\delta L)^L\right)^\Lambda \left(J\mathrm{d}^3x\right)^\Lambda, \tag{3.17a}$$

where the region $\mathscr{V}^\Lambda$ is composed of the points $\boldsymbol{x}^\Lambda$ mapped from $\mathscr{V}_0$ by $\boldsymbol{x}(=\boldsymbol{a}) \mapsto \boldsymbol{x}^\Lambda$. On dropping the Λ in the integrand, this becomes

$$\delta\mathscr{L} = \int_{\mathscr{V}^\Lambda} J(\delta L)^L\,\mathrm{d}^3x\,. \tag{3.17b}$$

Further use of (3.15*b*) and (3.16*b*) leads to the HEL form

$$J(\delta L)^L = \delta(JL^L) - J\nabla^L\boldsymbol{\cdot}(L^L\delta\boldsymbol{\xi}) \tag{3.17c}$$

of the integrand (cf. (2.15*c*)), in which

$$JL^L = \tfrac{1}{2}\sigma\left|\boldsymbol{v}^L\right|^2 - p^L\left((\sigma/\rho_0) - J\right) = Jp^L - \sigma\Pi^L\,. \tag{3.17d}$$

We formalize our procedures by identifying the variables on which our action (2.10) depends and write

$$(\mathscr{A}\ \equiv)\,A[\boldsymbol{u},\,\boldsymbol{\xi},\,\sigma,\,p^L] = \int_{t_0}^{t_1}\!\!\int_{\mathscr{V}^\Lambda} JL^L\,\mathrm{d}^3x\,\mathrm{d}t\,. \tag{3.18}$$

Two distinct formulations of Euler's equation emerge on varying only one of the virtual displacements $\delta\boldsymbol{\xi}$ and $\boldsymbol{\mu}$ that determine the HEL specification of the virtual displacement $\boldsymbol{\eta}^L$ of the material particle.

We begin with the choice

$$\boldsymbol{\mu} = \boldsymbol{0}, \qquad \text{for which} \qquad \delta\boldsymbol{u} = \boldsymbol{0}, \qquad \delta\sigma = 0 \tag{3.19a–c}$$

(see (3.12a, b)). With (3.13) this determines

$$\delta\boldsymbol{\xi} \;=\; \boldsymbol{\eta}^L\,. \tag{3.20}$$

In turn (3.14b) shows with the help of (3.7) that

$$\delta(\boldsymbol{v}^L) \;=\; \mathrm{D}_t\,\boldsymbol{\eta}^L \;=\; (\mathscr{D}_t\,\boldsymbol{\eta})^L \tag{3.21a}$$

similar to (2.14c) but with the ℓ replaced by L. Furthermore (3.16a, b) gives

$$\delta(p^L) \;=\; (\boldsymbol{\eta}\cdot\nabla p)^L \qquad \text{and} \qquad \delta J \;=\; J\,(\nabla\cdot\boldsymbol{\eta})^L\,. \tag{3.21b, c}$$

Armed with the increment values (3.19b, c) and (3.21), we may evaluate the action increment (3.17b), whose integrand (3.17c) becomes

$$\begin{aligned} J(\delta L)^L \;&=\; \sigma\boldsymbol{v}^L\cdot\delta(\boldsymbol{v}^L) - ((\sigma/\rho_0) - J)\delta(p^L) + p^L\delta J - J(\nabla\cdot(\boldsymbol{\eta}(p-\rho\Pi)))^L \\ &=\; -\sigma\,\boldsymbol{\eta}^L\cdot\boldsymbol{\mathscr{E}}^L + \sigma\big(\mathscr{D}_t(\boldsymbol{\eta}\cdot\boldsymbol{v})\big)^L + J\big(\nabla\cdot(\boldsymbol{\eta}\rho\Pi)\big)^L\,. \end{aligned} \tag{3.22}$$

On substituting $\sigma = J\rho^\Lambda$, dividing by J and then dropping the Λ, we recover the expression (2.23a) for δL obtained in §2.2. From a formal point of view, the separate application of the increments $\delta(p^L)$ and $\delta\boldsymbol{\xi} = \boldsymbol{\eta}^L$, as in (3.22), determines via (3.18) the variational derivatives

$$\frac{1}{J}\frac{\delta A}{\delta(p^L)} \;=\; \frac{\rho^L}{\rho_0} - 1 \qquad \text{and} \qquad -\,\frac{1}{\sigma}\frac{\delta A}{\delta\boldsymbol{\xi}} \;=\; \boldsymbol{\mathscr{E}}^L \tag{3.23a, b}$$

respectively. Their vanishing recovers

$$\rho^L \;=\; \rho_0 \qquad \text{and} \qquad \boldsymbol{\mathscr{E}}^L \;=\; \boldsymbol{0}\,. \tag{3.23c, d}$$

We consider next the choice

$$\delta\boldsymbol{\xi} \;=\; \boldsymbol{0}\,, \qquad \text{for which} \qquad \delta J \;=\; \boldsymbol{0}\,. \tag{3.24a, b}$$

With $\delta\boldsymbol{\xi} = \boldsymbol{0}$ the HEL displacement (3.13), velocity (3.14b) and pressure (3.16a) increments reduce to

$$\boldsymbol{\eta}^L \;=\; \boldsymbol{\mu}\cdot\nabla\boldsymbol{x}^L\,, \qquad \delta(\boldsymbol{v}^L) \;=\; \delta\boldsymbol{u}\cdot\nabla\boldsymbol{x}^L\,, \qquad \delta(p^L) \;=\; 0 \tag{3.25a–c}$$

respectively. Using these values and expression (3.17d) for $\mathscr{J}L^L$, the increment (3.17c) becomes

$$\begin{aligned} J(\delta L)^L \;=\; \delta(\mathscr{J}L^L) \;=\; -\delta(\sigma\Pi^L) \;&=\; \sigma\boldsymbol{v}^L\cdot\delta(\boldsymbol{v}^L) - \Pi^L\delta\sigma \\ &=\; \sigma\boldsymbol{V}\cdot\delta\boldsymbol{u} - \Pi^L\delta\sigma\,, \end{aligned} \tag{3.26a}$$

where

$$\boldsymbol{V} \;=\; (\nabla\boldsymbol{x}^L)\cdot\boldsymbol{v}^L \qquad \text{has the property} \qquad \boldsymbol{\mu}\cdot\boldsymbol{V} \;=\; \boldsymbol{\eta}^L\cdot\boldsymbol{v}^L\,. \tag{3.26b, c}$$

Then substitution of the values (3.12a, b) for $\delta\boldsymbol{u}$ and $\delta\sigma$ leads to

$$J(\delta L)^L = -\boldsymbol{\mu}\cdot\sigma\boldsymbol{E} + \sigma\mathrm{D}_t(\boldsymbol{\mu}\cdot\boldsymbol{V}) + \nabla\cdot(\boldsymbol{\mu}\sigma\Pi^L)\,, \tag{3.27a}$$

where

$$\boldsymbol{E} \;=\; \partial_t\boldsymbol{V} + \{\boldsymbol{u}\,,\,\boldsymbol{V}\} + \nabla\Pi^L\,. \tag{3.27b}$$

Furthermore, it may be shown directly that

$$\boldsymbol{E} \;=\; (\nabla\boldsymbol{x}^L)\cdot\boldsymbol{\mathscr{E}}^L \qquad \text{with the property} \qquad \boldsymbol{\mu}\cdot\boldsymbol{E} \;=\; \boldsymbol{\eta}^L\cdot\boldsymbol{\mathscr{E}}^L\,. \tag{3.27c, d}$$

In view of the relations (3.24a) to (3.26c), the terms $\delta(\mathscr{J}L^L)$, $\sigma\mathrm{D}_t(\boldsymbol{\mu}\cdot\boldsymbol{V})$ and $\nabla\cdot(\boldsymbol{\mu}\sigma\Pi^L)$ in (3.27a) are respectively equal to $J(\delta L)^L$, $\sigma\,(\mathscr{D}_t(\boldsymbol{\eta}\cdot\boldsymbol{v}))^L$ and $J\big(\nabla\cdot(\boldsymbol{\eta}\rho\Pi)\big)^L$ in (3.22).

In consequence, the remaining terms $-\sigma\boldsymbol{\eta}^L\cdot\boldsymbol{\mathscr{E}}^L$ and $-\boldsymbol{\mu}\cdot\sigma\boldsymbol{E}$ are necessarily equal too, consistent with (3.27*c*). By analogy with the Eulerian study of §2.2, our modified development leads to the HEL formulation

$$\boldsymbol{E} = \boldsymbol{0} \tag{3.28}$$

of Euler's equation. Evidently we may express (3.27*a*) as

$$\boldsymbol{E} = \frac{\partial}{\partial t}\left(\frac{1}{\sigma}\frac{\delta A}{\delta \boldsymbol{u}}\right) + \left\{\boldsymbol{u}\,,\left(\frac{1}{\sigma}\frac{\delta A}{\delta \boldsymbol{u}}\right)\right\} - \nabla\left(\frac{\delta A}{\delta\sigma}\right) \tag{3.29}$$

in terms of the functional derivatives

$$\frac{1}{\sigma}\frac{\delta A}{\delta \boldsymbol{u}} = \boldsymbol{V} = \boldsymbol{v}^L + \boldsymbol{V}^P\,, \qquad -\frac{\delta A}{\delta\sigma} = \Pi^L \equiv \frac{p^L}{\rho_0} - \tfrac{1}{2}|\boldsymbol{v}^L|^2 \tag{3.30a,b}$$

derived trivially from (3.26*a*) where, since $\boldsymbol{x}^L = \boldsymbol{x} + \boldsymbol{\xi}$ (see (3.4*b*)),

$$\boldsymbol{V}^P = (\nabla\boldsymbol{\xi})\cdot\boldsymbol{v}^L\,. \tag{3.30c}$$

The quantity $\rho_0\boldsymbol{V}^P$ is often referred to as the pseudo-momentum at $\boldsymbol{x}^L$.

Finally, since

$$\mathrm{D}_t\,(\boldsymbol{V}\cdot\mathrm{d}\boldsymbol{x}) = \big(\partial_t\boldsymbol{V} + \{\boldsymbol{u}\,,\,\boldsymbol{V}\}\big)\cdot\mathrm{d}\boldsymbol{x} \tag{3.31a}$$

(cf. (2.28)), Kelvin's circulation theorem (2.29*b*) may be expressed in the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\oint_{\mathscr{C}^A}\boldsymbol{V}\cdot\mathrm{d}\boldsymbol{x} = \boldsymbol{0} \tag{3.31b}$$

for closed curves $\mathscr{C}^A$ composed of points $\boldsymbol{x}$, that move with the reference velocity $\boldsymbol{u}(\boldsymbol{x},t)$. We stress that the HEL Euler's equation $\boldsymbol{E} = \boldsymbol{0}$ (see (3.28)) concerns the state of the flow at the displaced points $\boldsymbol{x}^L$ rather than $\boldsymbol{x}$. This means that (3.31*b*) is simply a reiteration of (2.29*b*) evaluated for circuits $\mathscr{C}^\ell$ composed of points $\boldsymbol{x}^L$ moving with velocity $\boldsymbol{v}^L$, i.e.

$$\oint_{\mathscr{C}^\ell}\boldsymbol{v}^L\cdot\mathrm{d}\boldsymbol{x}^L = \oint_{\mathscr{C}^A}\boldsymbol{V}\cdot\mathrm{d}\boldsymbol{x}\,. \tag{3.31c}$$

3.2. *An Eulerian representation*

The usage of the HEL representation is generally restricted to small $|\boldsymbol{\xi}|$, as we explain in §5 below. So using the Taylor series results $\boldsymbol{v} = \boldsymbol{v}^L - \boldsymbol{\xi}\cdot\nabla\boldsymbol{v}^L + O(|\boldsymbol{\xi}|^2)$ and $\rho = \rho^L - \boldsymbol{\xi}\cdot\nabla\rho^L + O(|\boldsymbol{\xi}|^2)$ (derivable from (5.1) below), we see that the flow has the Eulerian representation $\boldsymbol{v} = \boldsymbol{u} + (\partial_t\boldsymbol{\xi} + [\boldsymbol{u},\boldsymbol{\xi}]) + O(|\boldsymbol{\xi}|^2)$ and $\rho = \sigma - \nabla\cdot(\sigma\boldsymbol{\xi}) + O(|\boldsymbol{\xi}|^2)$ (derivable from (3.6*b*) and (3.8*b*) but cf. (2.21*a, b*)). To take advantage of the HEL representation in an Eulerian setting, Holm (2002*b*) has proposed replacing $\boldsymbol{u}$, σ and $\boldsymbol{\xi}$ by new variables $\overline{\boldsymbol{v}}(\boldsymbol{x},t)$, $\overline{\rho}(\boldsymbol{x},t)$ and $\boldsymbol{\zeta}(\boldsymbol{x},t)$, which differ from them only at $O(|\boldsymbol{\xi}|^2)$ (see (5.3*a, c*) below), such that the relations,

$$\boldsymbol{v} = \overline{\boldsymbol{v}} + \boldsymbol{v}' \qquad \text{and} \qquad \rho = \overline{\rho} + \rho' \tag{3.32a,b}$$

with

$$\boldsymbol{v}' = \partial_t\boldsymbol{\zeta} + [\overline{\boldsymbol{v}}\,,\,\boldsymbol{\zeta}] \qquad \text{and} \qquad \rho' = -\nabla\cdot(\overline{\rho}\,\boldsymbol{\zeta})\,, \tag{3.32c,d}$$

are exact. At this stage the overbar does not mean the average value that it will in later sections. So for the moment $\overline{\boldsymbol{v}}$ is arbitrary and not the average value of $\boldsymbol{v}$. This also implies that the decomposition (3.32) is not unique as we are free to choose one or other of $\overline{\boldsymbol{v}}$ and $\boldsymbol{\zeta}$ at our convenience, just as we do for $\boldsymbol{u}$ and $\boldsymbol{\xi}$ in §3.1. To stress that the arguments $\boldsymbol{v} = \boldsymbol{v}(\overline{\boldsymbol{v}}, \boldsymbol{\zeta})$ and $\rho = \rho(\overline{\rho}, \boldsymbol{\zeta})$ of the action $\mathscr{A}[\boldsymbol{v}, \rho, p]$, whose variation is given by (2.20*a*), vary parametrically in response to variations of $\overline{\boldsymbol{v}}$, $\overline{\rho}$ and $\boldsymbol{\zeta}$, we introduce the notation

$$A^E[\overline{\boldsymbol{v}}, \overline{\rho}, \boldsymbol{\zeta}, p] \equiv \mathscr{A}[\boldsymbol{v}(\overline{\boldsymbol{v}}, \boldsymbol{\zeta}), \rho(\overline{\rho}, \boldsymbol{\zeta}), p]. \tag{3.33}$$

Because the fluid has constant density $\rho = \rho_0$ (a result recovered by the vanishing of the p variation as in (2.27)), it is natural to assume that $\overline{\rho} = \rho_0$ together with $\nabla\boldsymbol{\cdot}\overline{\boldsymbol{v}} = 0$. It follows that

$$\rho' = 0 \qquad \text{and} \qquad \nabla\boldsymbol{\cdot}\boldsymbol{\zeta} = 0 \tag{3.34a, b}$$

from (3.32*d*). Then, with the help of the identity $\nabla\boldsymbol{\cdot}[\overline{\boldsymbol{v}}, \boldsymbol{\zeta}] = \nabla\boldsymbol{\cdot}(\overline{\boldsymbol{v}}\,\nabla\boldsymbol{\cdot}\boldsymbol{\zeta} - \boldsymbol{\zeta}\,\nabla\boldsymbol{\cdot}\overline{\boldsymbol{v}})$ (see Appendix A, (A1*b*)), the divergence of (3.32*c*) shows that

$$\nabla\boldsymbol{\cdot}\boldsymbol{v}' = 0, \qquad \text{since} \qquad \nabla\boldsymbol{\cdot}\overline{\boldsymbol{v}} = 0. \tag{3.34c, d}$$

As in §2.2, the results (3.34) cannot be used until the calculation of the required variational derivatives is complete.

Consider the virtual increments $\delta\boldsymbol{v}$ and $\delta\rho$ of the velocity $\boldsymbol{v} = \boldsymbol{v}(\overline{\boldsymbol{v}}, \boldsymbol{\zeta})$ and the density $\rho = \rho(\overline{\rho}, \boldsymbol{\zeta})$ caused by virtual displacement of the particle paths. In terms of the increments $\delta\overline{\boldsymbol{v}}$, $\delta\overline{\rho}$ and $\delta\boldsymbol{\zeta}$, the flow composition (3.32) enables us to express the result as

$$\delta\boldsymbol{v} = \delta\overline{\boldsymbol{v}} + \delta\boldsymbol{v}' \qquad \text{and} \qquad \delta\rho = \delta\overline{\rho} + \delta\rho', \tag{3.35a, b}$$

where

$$\delta\boldsymbol{v}' = \partial_t\,\delta\boldsymbol{\zeta} + [\overline{\boldsymbol{v}}, \delta\boldsymbol{\zeta}] + [\delta\overline{\boldsymbol{v}}, \boldsymbol{\zeta}] \qquad \text{and} \qquad \delta\rho' = -\nabla\boldsymbol{\cdot}(\overline{\rho}\,\delta\boldsymbol{\zeta}) - \nabla\boldsymbol{\cdot}(\delta\overline{\rho}\,\boldsymbol{\zeta}). \tag{3.35c, d}$$

Note the similarity of the Eulerian expression (3.35*c*) for $\delta\boldsymbol{v}'$ with the corresponding HEL form $\delta(\boldsymbol{v}^L) - \delta\boldsymbol{u} = \mathrm{D}_t\,\delta\boldsymbol{\xi} + \delta\boldsymbol{u}\boldsymbol{\cdot}\nabla\boldsymbol{\xi}$ obtained from (3.14*b*). Thus the increment

$$\delta A^E = \int_{t_0}^{t_1}\int_{\mathscr{V}\ell} \delta L\; \mathrm{d}^3x\; \mathrm{d}t \tag{3.36a}$$

of the action (3.33) (see (2.15*d*)) with $\delta L = \rho\boldsymbol{v}\boldsymbol{\cdot}\delta\boldsymbol{v} - \Pi\,\delta\rho$ (see (2.20*c*)) determined by the virtual increments (3.35) is given by

$$\delta L = \rho\boldsymbol{v}\boldsymbol{\cdot}(\delta\overline{\boldsymbol{v}} + [\delta\overline{\boldsymbol{v}}, \boldsymbol{\zeta}] + \partial_t\,\delta\boldsymbol{\zeta} + [\overline{\boldsymbol{v}}, \delta\boldsymbol{\zeta}]) - \Pi(\delta\overline{\rho} - \nabla\boldsymbol{\cdot}(\delta\overline{\rho}\,\boldsymbol{\zeta}) - \nabla\boldsymbol{\cdot}(\overline{\rho}\,\delta\boldsymbol{\zeta})). \tag{3.36b}$$

It may be expressed in the usual way as

$$\begin{aligned} \delta L = {} & \left(\frac{\delta A^E}{\delta\overline{\boldsymbol{v}}}\right)\boldsymbol{\cdot}\delta\overline{\boldsymbol{v}} + \left(\frac{\delta A^E}{\delta\overline{\rho}}\right)\delta\overline{\rho} + \left(\frac{\delta A^E}{\delta\boldsymbol{\zeta}}\right)\boldsymbol{\cdot}\delta\boldsymbol{\zeta} \\ & + \partial_t(\rho\boldsymbol{v}\boldsymbol{\cdot}\delta\boldsymbol{\zeta}) + \nabla\boldsymbol{\cdot}\Big(-\boldsymbol{\zeta}(\rho\boldsymbol{v}\boldsymbol{\cdot}\delta\overline{\boldsymbol{v}}) + \Pi\,\boldsymbol{\zeta}\,\delta\overline{\rho} + \overline{\boldsymbol{v}}(\rho\boldsymbol{v}\boldsymbol{\cdot}\delta\boldsymbol{\zeta}) + \overline{\rho}\Pi\,\delta\boldsymbol{\zeta}\Big), \end{aligned} \tag{3.37a}$$

where, in terms of the $\langle\,\cdot\,,\,\cdot\,\rangle$ notation introduced in Appendix A and defined by (A2*b*),

$$\frac{\delta A^E}{\delta\overline{\boldsymbol{v}}} = \rho\boldsymbol{v} + \langle\boldsymbol{\zeta}, \rho\boldsymbol{v}\rangle, \qquad -\frac{\delta A^E}{\delta\overline{\rho}} = \Pi + \boldsymbol{\zeta}\boldsymbol{\cdot}\nabla\Pi, \tag{3.37b, c}$$

$$-\frac{\delta A^E}{\delta\boldsymbol{\zeta}} = \partial_t(\rho\boldsymbol{v}) + \langle\overline{\boldsymbol{v}}, \rho\boldsymbol{v}\rangle + \overline{\rho}\,\nabla\Pi. \tag{3.37d}$$

Some care must be taken in our interpretation of the definitions (3.37*b–d*) as the incrementation of L with respect to the separate variations of $\overline{\boldsymbol{v}}$, $\overline{\rho}$ and $\boldsymbol{\zeta}$ leads to extra terms that do not vanish on application of the boundary conditions implied by Hamilton's principle. For example, varying $\overline{\boldsymbol{v}}$ keeping $\overline{\rho}$ and $\boldsymbol{\zeta}$ fixed gives $\delta L =$

$(\delta A^E/\delta\overline{\boldsymbol{v}})\cdot\delta\overline{\boldsymbol{v}} - \nabla\cdot(\boldsymbol{\zeta}(\rho\boldsymbol{v}\cdot\delta\overline{\boldsymbol{v}}))$ and on application of the divergence theorem the surface integral of the normal component of $\boldsymbol{\zeta}(\rho\boldsymbol{v}\cdot\delta\overline{\boldsymbol{v}})$ does not in general vanish. The same remark applies to the increment of δL caused by the separate variations of $\overline{\rho}$ and $\boldsymbol{\zeta}$. This is an unattractive feature of our Eulerian presentation, which did not happen when $\boldsymbol{u}$ and σ were varied separately in the HEL development of §3.1. Nevertheless, once the increments $\delta\overline{\boldsymbol{v}}$, $\delta\overline{\rho}$ and $\delta\boldsymbol{\zeta}$ have been correctly related to the virtual path displacement $\boldsymbol{\eta}$, we are able in Appendix B to show that the combined surface contribution as well as the initial $t = t_0$ and final $t = t_1$ time contributions, all identified by (3.37*a*), do indeed vanish. This result validates our key formula (3.43) below for $\boldsymbol{\mathscr{E}}$.

Let us first consider the special case $\boldsymbol{\zeta} = \boldsymbol{0}$, for which $\boldsymbol{v}' = \boldsymbol{0}$, $\rho' = 0$ and in consequence $\overline{\boldsymbol{v}} = \boldsymbol{v}$, $\overline{\rho} = \rho$. When we apply the virtual path increment $\boldsymbol{\eta}$, we recover the development of §2.2 upon setting $\delta\overline{\boldsymbol{v}} = \boldsymbol{0}$, $\delta\overline{\rho} = 0$ and $\delta\boldsymbol{\zeta} = \boldsymbol{\eta}$. Then (3.37*a*) reduces to (2.25*a*), in which

$$\rho\boldsymbol{\mathscr{E}} = -\left.\frac{\delta A^E}{\delta\boldsymbol{\zeta}}\right|_{\boldsymbol{\zeta}=\boldsymbol{0}} \tag{3.38}$$

(cf. (2.25*b*) with (3.37*d*)). Of course, when $\boldsymbol{\zeta} = \boldsymbol{0}$ the values of $\delta A^E/\delta\overline{\boldsymbol{v}}$ and $\delta A^E/\delta\overline{\rho}$ defined by (3.37*b, c*) are equal to the values of $\delta A/\delta\boldsymbol{v}$ and $\delta A/\delta\rho$ defined by (2.26*b, c*). From a formal point of view, if we adopt the alternative strategy of setting $\delta\boldsymbol{\zeta} = \boldsymbol{0}$ so that $\delta\boldsymbol{v}' = \boldsymbol{0}$ and $\delta\rho' = 0$ and then invoke the representations (2.21*a, b*) for $\delta\overline{\boldsymbol{v}}$ and $\delta\overline{\rho}$, we recover Euler's equation in the form $\boldsymbol{\mathscr{E}} = \boldsymbol{0}$ based on (2.26*a*).

For the general case $\boldsymbol{\zeta} \neq \boldsymbol{0}$, which is the concern of this section, we have $\boldsymbol{v}' \neq \boldsymbol{0}$ and $\overline{\boldsymbol{v}} \neq \boldsymbol{v}$. Then the attractive result (3.38) no longer holds. Thus to apply Hamilton's principle, we need to link $\delta\overline{\boldsymbol{v}}$, $\delta\boldsymbol{v}'$ and $\delta\boldsymbol{\zeta}$ to $\boldsymbol{\eta}$ in some non-trivial way. To that end we note that just as the decomposition $\boldsymbol{v} = \overline{\boldsymbol{v}} + \boldsymbol{v}'$ is not unique, neither is the decomposition $\delta\boldsymbol{v} = \delta\overline{\boldsymbol{v}} + \delta\boldsymbol{v}'$, as our discussion above of the special case $\boldsymbol{\zeta} = \boldsymbol{0}$ emphasized. Since $\delta\boldsymbol{v}'$ is determined by $\delta\overline{\boldsymbol{v}}$ and $\delta\boldsymbol{\zeta}$, the various possible dependences of $\delta\overline{\boldsymbol{v}}$ and $\delta\boldsymbol{\zeta}$ on $\boldsymbol{\eta}$ need to be explored.

Guided by the HEL choice $\delta\boldsymbol{u} = \boldsymbol{0}$ (see (3.19*b*)) of §3.1, which has the useful consequence $\delta\boldsymbol{\xi} = \boldsymbol{\eta}^L$ (see (3.20)), we consider the implications of setting $\delta\overline{\boldsymbol{v}} = \boldsymbol{0}$ with $\delta\boldsymbol{v} = \delta\boldsymbol{v}'$. From (2.21*a*) and (3.35*a, c*) it follows that

$$\partial_t(\delta\boldsymbol{\zeta} - \boldsymbol{\eta}) + [\overline{\boldsymbol{v}}, (\delta\boldsymbol{\zeta} - \boldsymbol{\eta})] = [\boldsymbol{v}', \boldsymbol{\eta}] \qquad (\delta\overline{\boldsymbol{v}} = \boldsymbol{0}). \tag{3.39}$$

So when $\boldsymbol{v}' \neq \boldsymbol{0}$, we no longer have the simple solution $\delta\boldsymbol{\zeta} = \boldsymbol{\eta}$ adopted for the special $\boldsymbol{\zeta} = \boldsymbol{0}$ case and in this important respect our development cannot parallel the HEL analysis of §3.1 based on $\delta\boldsymbol{\xi} = \boldsymbol{\eta}^L$. Instead our choice $\delta\overline{\boldsymbol{v}} = \boldsymbol{0}$ necessitates solving the inhomogeneous equation (3.39) for the difference vector $\delta\boldsymbol{\zeta} - \boldsymbol{\eta}$. This leads to a non-local relationship between $\delta\boldsymbol{\zeta}(\boldsymbol{x}, t)$ and the given displacement field $\boldsymbol{\eta}(\boldsymbol{x}, t)$, which is inconvenient for the application of Hamilton's principle.

Consider now the general case $\delta\overline{\boldsymbol{v}} \neq \boldsymbol{0}$. As in our derivation of (3.39), we equate the expressions for $\delta\boldsymbol{v}$ given by both (2.21*a*) and (3.35*a, c*). After some algebra, which involves use of the identity (A1*c*) in Appendix A and the help of (3.32*c*), we may derive

$$\partial_t\boldsymbol{\varpi} + [\overline{\boldsymbol{v}}, \boldsymbol{\varpi}] = \delta\overline{\boldsymbol{v}} - (\partial_t\boldsymbol{\eta} + [\overline{\boldsymbol{v}}, \boldsymbol{\eta}]) - [\boldsymbol{\zeta}, (\delta\overline{\boldsymbol{v}} - (\partial_t\boldsymbol{\eta} + [\overline{\boldsymbol{v}}, \boldsymbol{\eta}]))], \tag{3.40a}$$

where

$$\boldsymbol{\varpi} = [\boldsymbol{\zeta}, \boldsymbol{\eta}] - \delta\boldsymbol{\zeta}. \tag{3.40b}$$

A similar calculation, which equates the expressions for $\delta\rho$ given by both (2.21*b*) and (3.35*b*, *d*), leads with the help of (3.32*d*) to

$$-\nabla\cdot(\overline{\rho}\,\boldsymbol{\varpi}) = \delta\overline{\rho} + \nabla\cdot(\overline{\rho}\,\boldsymbol{\eta}) - \nabla\cdot(\boldsymbol{\zeta}(\delta\overline{\rho} + \nabla\cdot(\overline{\rho}\,\boldsymbol{\eta}))). \tag{3.40c}$$

Clearly the choice

$$\delta\overline{\boldsymbol{v}} = \partial_t\boldsymbol{\eta} + [\overline{\boldsymbol{v}}\,,\,\boldsymbol{\eta}] \qquad \text{and} \qquad \delta\overline{\rho} = -\nabla\cdot(\overline{\rho}\,\boldsymbol{\eta})\,, \tag{3.41a, b}$$

which is similar to (2.21*a*, *b*) but with $\boldsymbol{v}$ and ρ replaced by $\overline{\boldsymbol{v}}$ and $\overline{\rho}$ respectively, ensures that the right-hand sides of both (3.40*a*) and (3.40*c*) vanish. This fortuitous situation means that what is left of (3.40*a*, *c*) has the trivial solution $\boldsymbol{\varpi} = \mathbf{0}$:

$$\delta\boldsymbol{\zeta} = [\boldsymbol{\zeta}\,,\,\boldsymbol{\eta}]\,. \tag{3.42}$$

This choice, which we now adopt, provides a natural way of defining the increment $\delta\boldsymbol{\zeta}$ in terms of the given virtual displacement $\boldsymbol{\eta}$ so as to ensure that the velocity and density increments $\delta\boldsymbol{v}$ and $\delta\rho$ (see (3.35*a*, *b*)) are determined correctly by (3.41) and (3.35*c*, *d*). That is all we need. Any boundary conditions on the virtual displacement $\boldsymbol{\eta}$ such as its vanishing at the initial and final times must, of course, be adhered to but there are no further independent boundary conditions on $\delta\boldsymbol{\zeta}$. So for example, there is no requirement that $\delta\boldsymbol{\zeta}$ vanishes at the initial and final times. Instead $\delta\boldsymbol{\zeta}$ simply takes the value (generally non-zero) that (3.42) dictates.

Upon substitution of the values of $\delta\overline{\boldsymbol{v}}$, $\delta\overline{\rho}$ and $\delta\boldsymbol{\zeta}$ given by (3.41*a*, *b*) and (3.42) into (3.37*a*), we may recover the expression (2.25*a*) for δL, in which $\rho\boldsymbol{\mathscr{E}}$ is now given by

$$\rho\boldsymbol{\mathscr{E}} = \frac{\partial}{\partial t}\left(\frac{\delta A^E}{\delta\overline{\boldsymbol{v}}}\right) + \left\langle \overline{\boldsymbol{v}}\,,\,\frac{\delta A^E}{\delta\overline{\boldsymbol{v}}}\right\rangle + \left\langle \boldsymbol{\zeta}\,,\,\frac{\delta A^E}{\delta\boldsymbol{\zeta}}\right\rangle - \overline{\rho}\,\nabla\left(\frac{\delta A^E}{\delta\overline{\rho}}\right). \tag{3.43}$$

The verification of this claim involves some lengthy algebra, which we outline briefly in Appendix B.

So far all the variational derivatives have been calculated on the basis that compressible variations are admissible. Application of Hamilton's principle as in §2.2 recovers $\boldsymbol{\mathscr{E}} = \mathbf{0}$, where now (3.43) is evaluated for the incompressible unperturbed state $\rho = \overline{\rho} = \rho_0$ with $\nabla\cdot\boldsymbol{\zeta} = 0$, $\nabla\cdot\overline{\boldsymbol{v}} = 0$ (see (3.34*b*, *d*)). Then, in view of (A2*c*) in Appendix A, all our $\langle\,\cdot\,,\,\cdot\,\rangle$ operations reduce to $\{\,\cdot\,,\,\cdot\,\}$ operations and in consequence (3.43) leads to the new variational form

$$\boldsymbol{\mathscr{E}} \equiv \frac{\partial}{\partial t}\left(\frac{1}{\overline{\rho}}\frac{\delta A^E}{\delta\overline{\boldsymbol{v}}}\right) + \left\{\overline{\boldsymbol{v}}\,,\left(\frac{1}{\overline{\rho}}\frac{\delta A^E}{\delta\overline{\boldsymbol{v}}}\right)\right\} + \left\{\boldsymbol{\zeta}\,,\left(\frac{1}{\overline{\rho}}\frac{\delta A^E}{\delta\boldsymbol{\zeta}}\right)\right\} - \nabla\left(\frac{\delta A^E}{\delta\overline{\rho}}\right) = \mathbf{0} \tag{3.44}$$

of Euler's equation, in which (3.37*b*–*d*) reduce to

$$\frac{1}{\overline{\rho}}\frac{\delta A^E}{\delta\overline{\boldsymbol{v}}} = \boldsymbol{V}^E \equiv \boldsymbol{v} + \{\boldsymbol{\zeta}\,,\,\boldsymbol{v}\}\,, \qquad -\frac{\delta A^E}{\delta\overline{\rho}} = \Pi^E \equiv \Pi + \boldsymbol{\zeta}\cdot\nabla\,\Pi\,, \tag{3.45a, b}$$

$$-\frac{1}{\overline{\rho}}\frac{\delta A^E}{\delta\boldsymbol{\zeta}} = \partial_t\boldsymbol{v} + \{\overline{\boldsymbol{v}}\,,\,\boldsymbol{v}\} + \nabla\Pi\,. \tag{3.45c}$$

Equation (3.44) may be compared with the fourth bullet point equation on p. 268 of Holm (2002*b*), which he refers to as 'the Euler-Poincaré equation' (EP equation). This omits our penultimate term in (3.44) although, since $\boldsymbol{\zeta} \neq \mathbf{0}$, it is not generally zero:

$$\left.\left\{\boldsymbol{\zeta}\,,\left(\frac{1}{\overline{\rho}}\frac{\delta A^E}{\delta\boldsymbol{\zeta}}\right)\right\}\right|_{\boldsymbol{\zeta}\neq\mathbf{0}} \neq \mathbf{0}\,. \tag{3.46}$$

The point that we wish to emphasize is that at $\boldsymbol{\zeta} = \mathbf{0}$ we have two distinct variational derivative forms (3.38) and (2.26*a*) for $\boldsymbol{\mathscr{E}}$, which may be used in Euler's equation

$\boldsymbol{\mathscr{E}} = \mathbf{0}$. This is in contrast with the case $\boldsymbol{\zeta} \neq \mathbf{0}$. Then we only have the one form (3.44), in which the additional term (3.46) appears that cannot be ignored. Put another way, the form of (3.44) tells us that we cannot vary the action integral by incrementing $\overline{\boldsymbol{v}}$ alone. For if we did and held $\boldsymbol{\zeta}$ fixed, the velocity increment $\delta\boldsymbol{v}$ would not simply be $\delta\overline{\boldsymbol{v}}$ but contain the additional contribution $\delta\boldsymbol{v}' = [\delta\overline{\boldsymbol{v}}\,,\,\boldsymbol{\zeta}]$ (see (3.35*c*)). To take proper account of the virtual path displacement, we must simultaneously increment $\boldsymbol{\zeta}$ as specified by (3.42); Holm (2002*b*) does not do that.

Continuing with the case $\boldsymbol{\zeta} \neq \mathbf{0}$, we rewrite (3.45*c*) in the form

$$-\frac{1}{\overline{\rho}}\frac{\delta A^E}{\delta\boldsymbol{\zeta}} = \boldsymbol{\mathscr{E}} - \{\boldsymbol{v}'\,,\,\boldsymbol{v}\}\,, \tag{3.47}$$

where we have employed the representation (2.23*b*) of $\boldsymbol{\mathscr{E}}$. The substitution of this value of $\delta A^E/\delta\boldsymbol{\zeta}$ into (3.44) suggests that we should construct

$$\boldsymbol{E}^E \equiv \boldsymbol{\mathscr{E}} + \{\boldsymbol{\zeta}\,,\,\boldsymbol{\mathscr{E}}\}\,. \tag{3.48a}$$

Then upon substitution for the remaining variational derivatives $\delta A^E/\delta\overline{\boldsymbol{v}}$ and $\delta A^E/\delta\overline{\rho}$ (see (3.45*a*, *b*)), we obtain

$$\boldsymbol{E}^E \equiv \partial_t\boldsymbol{V}^E + \left\{\overline{\boldsymbol{v}}\,,\,\boldsymbol{V}^E\right\} + \{\boldsymbol{\zeta}\,,\{\boldsymbol{v}'\,,\,\boldsymbol{v}\}\} + \nabla\Pi^E\,. \tag{3.48b}$$

Since $\boldsymbol{\mathscr{E}} = \mathbf{0}$, this provides a new form,

$$\boldsymbol{E}^E = \mathbf{0}\,, \tag{3.48c}$$

of Euler's equation. Significantly, the omission on p. 268 of Holm (2002*b*), noted above, is perpetuated in his Section 4.6 on p. 275. There, his EP equation halfway down the page omits our term $\{\boldsymbol{\zeta}\,,\{\boldsymbol{v}'\,,\,\boldsymbol{v}\}\}$ but is otherwise identical to (3.48*b*), at any rate after averaging as in our (4.7*a*) below. It should be stressed that our equations (3.44) and (3.48*b*, *c*) are exact and do not assume that $\boldsymbol{\zeta}$ is small. Of course, (3.48*b*, *c*) may be derived directly from Euler's equation (2.1*a*) without any appeal to Hamilton's principle.

We introduce the notion of the material derivative

$$\mathrm{D}_t^E \equiv \partial_t + \overline{\boldsymbol{v}}\boldsymbol{\cdot}\nabla \tag{3.49a}$$

following points moving with velocity $\overline{\boldsymbol{v}}$ for which

$$\mathrm{D}_t^E\left(\boldsymbol{V}^E\boldsymbol{\cdot}\mathrm{d}\boldsymbol{x}\right) = \left(\partial_t\boldsymbol{V}^E + \{\overline{\boldsymbol{v}}\,,\,\boldsymbol{V}^E\}\right)\boldsymbol{\cdot}\mathrm{d}\boldsymbol{x}\,. \tag{3.49b}$$

Following the arguments at the end of §2, Kelvin's circulation theorem (2.29*b*) becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}\oint_{\mathscr{C}^E}\boldsymbol{V}^E\boldsymbol{\cdot}\mathrm{d}\boldsymbol{x} = -\oint_{\mathscr{C}^E}\{\boldsymbol{\zeta}\,,\{\boldsymbol{v}'\,,\,\boldsymbol{v}\}\}\boldsymbol{\cdot}\mathrm{d}\boldsymbol{x} \tag{3.50}$$

for closed curves $\mathscr{C}^E$ composed of points $\boldsymbol{x}$, that move with velocity $\overline{\boldsymbol{v}}(\boldsymbol{x}, t)$.

For our later developments, we follow Holm (2002*b*) and express the difference

$$\boldsymbol{V}^E - \boldsymbol{v} = \{\boldsymbol{\zeta}\,,\,\boldsymbol{v}\}\,(= \boldsymbol{\zeta}\boldsymbol{\cdot}\nabla\boldsymbol{v} + (\nabla\boldsymbol{\zeta})\boldsymbol{\cdot}\boldsymbol{v}) = \boldsymbol{v}^s + \boldsymbol{V}^p \tag{3.51a}$$

(see (3.45*a*)), as the sum of the constituent parts

$$\boldsymbol{v}^s = \boldsymbol{\zeta}\boldsymbol{\cdot}\nabla\,\boldsymbol{v}\,, \qquad \boldsymbol{V}^p = (\nabla\,\boldsymbol{\zeta})\boldsymbol{\cdot}\boldsymbol{v} \tag{3.51b, c}$$

of $\{\boldsymbol{\zeta}\,,\,\boldsymbol{v}\}$. The superscripts s and p are used to signify that $\boldsymbol{v}^s$ and $\rho_0\boldsymbol{V}^p$ are parts of the Stokes drift $\boldsymbol{v}^S$ (see (4.3*a*) below) and pseudo-momentum $\rho_0\boldsymbol{V}^P$ (see (4.4) below).

The remainder of our paper concerns the relation between Euler's equation $\boldsymbol{E}^E = \mathbf{0}$ involving the advective velocity $\overline{\boldsymbol{v}}$ (see (3.48*b*, *c*)) and the previous HEL version

$\boldsymbol{E} = \boldsymbol{0}$ involving the advective velocity $\boldsymbol{u}$ (see (3.27*b*) and (3.28)). In preparation for that discussion, we introduce $\overline{\boldsymbol{v}}$ into the HEL version and write

$$\boldsymbol{E} \equiv \partial_t \boldsymbol{V} + \{\overline{\boldsymbol{v}}, \boldsymbol{V}\} + \left\{\overline{\boldsymbol{v}^S}, \boldsymbol{V}\right\} + \nabla \Pi^L = \boldsymbol{0}, \tag{3.52a}$$

where $\overline{\boldsymbol{v}^S}$ is the velocity difference

$$\overline{\boldsymbol{v}^S} = \boldsymbol{u} - \overline{\boldsymbol{v}} \qquad \text{satisfying} \qquad \nabla\cdot\boldsymbol{u} = \nabla\cdot\overline{\boldsymbol{v}^S} \tag{3.52b, c}$$

since $\nabla\cdot\overline{\boldsymbol{v}} = 0$ (see (3.34*d*)). Thus a form of Kelvin's circulation theorem involving the convective velocity $\overline{\boldsymbol{v}}$ similar to (3.50) is

$$\frac{\mathrm{d}}{\mathrm{d}t} \oint_{\mathscr{C}^E} \boldsymbol{V}\cdot \mathrm{d}\boldsymbol{x} = - \oint_{\mathscr{C}^E} \left\{\overline{\boldsymbol{v}^S}, \boldsymbol{V}\right\}\cdot \mathrm{d}\boldsymbol{x}\,. \tag{3.53}$$

The relevance of this equation becomes apparent after averages are taken in the next section and $\overline{\boldsymbol{v}^S}$ turns out to be the Stokes drift velocity (see (4.1*d*) and (4.3*a*)).

4. Averages

The flow decompositions set up in the previous section provide frameworks for analysing flow fluctuations $\boldsymbol{v}'$ superimposed on a basic mean flow $\overline{\boldsymbol{v}}$. Henceforth the overbar denotes that an average is taken, while the prime denotes the remaining fluctuating part. So on writing $\boldsymbol{v}' = \boldsymbol{v} - \overline{\boldsymbol{v}}$ we have $\overline{\boldsymbol{v}'} = \partial_t \overline{\boldsymbol{\zeta}} + [\overline{\boldsymbol{v}}, \overline{\boldsymbol{\zeta}}] = \boldsymbol{0}$ (see (3.32*c*)), a condition that is met when $\overline{\boldsymbol{\zeta}} = \boldsymbol{0}$. In the same spirit we demand in the Lagrangian representation that $\boldsymbol{u}$ has no fluctuating part so that $\overline{\boldsymbol{u}} = \boldsymbol{u}$, while $\boldsymbol{\xi}$ has no mean part $\overline{\boldsymbol{\xi}} = \boldsymbol{0}$. The immediate consequence of these assumptions is that the material derivatives D_t and D_t^E, defined by (3.3*b*) and (3.49*a*) respectively, relate to mean velocities. It then follows from (3.6*b*) that $\overline{\boldsymbol{v}^L} = \boldsymbol{u}$ and $\overline{\mathrm{D}_t\boldsymbol{\xi}} = \boldsymbol{0}$. In summary our basic assumptions are

$$\overline{\boldsymbol{\xi}} = \overline{\boldsymbol{\zeta}} = \boldsymbol{0} \qquad \text{and} \qquad \overline{\mathrm{D}_t\boldsymbol{\xi}} = \boldsymbol{0}, \qquad \overline{\mathrm{D}_t^E\boldsymbol{\zeta}} = \boldsymbol{0}, \tag{4.1a–c}$$

where

$$\boldsymbol{u} = \overline{\boldsymbol{v}^L}\,. \tag{4.1d}$$

It should be pointed out however that $\overline{\boldsymbol{v}}$ and $\boldsymbol{u}$ are unequal. More generally the average of the material derivatives of ϕ^L (see (3.5*a*)) and ϕ satisfy

$$\overline{\mathrm{D}_t\phi^L} = \mathrm{D}_t\,\overline{\phi^L}\,, \qquad \overline{\mathrm{D}_t^E\phi} = \mathrm{D}_t^E\,\overline{\phi}\,. \tag{4.2}$$

We define a new vector $\boldsymbol{v}^\dagger$ via the velocity difference

$$\boldsymbol{v}^S \equiv \boldsymbol{v}^L - \boldsymbol{v} = \boldsymbol{v}^s + \boldsymbol{v}^\dagger\,, \tag{4.3a}$$

where $\boldsymbol{v}^s$ is given by (3.51*b*). Since $\overline{\boldsymbol{v}^L} = \boldsymbol{u}$, the average $\overline{\boldsymbol{v}^S} = \overline{\boldsymbol{v}^L} - \overline{\boldsymbol{v}}$ is consistent with (3.52*b*) and called the Stokes drift. We note that the fluctuating part of $\boldsymbol{v}^S$ is

$$\boldsymbol{v}^{S'} \equiv \boldsymbol{v}^S - \overline{\boldsymbol{v}^S} = \partial_t(\boldsymbol{\xi} - \boldsymbol{\zeta}) + [\overline{\boldsymbol{v}}, (\boldsymbol{\xi} - \boldsymbol{\zeta})] + \boldsymbol{\xi}\cdot\nabla\overline{\boldsymbol{v}^L}\,, \tag{4.3b}$$

which serves to emphasize that $\boldsymbol{\xi}$ and $\boldsymbol{\zeta}$ are distinct vectors. In the spirit of (4.3*a*), we also introduce $\boldsymbol{V}^\dagger$ defined by

$$\boldsymbol{V}^P = \boldsymbol{V} - \boldsymbol{v}^L = \boldsymbol{V}^p + \boldsymbol{V}^\dagger\,, \tag{4.4}$$

where $\boldsymbol{V}$, $\boldsymbol{V}^P$ and $\boldsymbol{V}^p$ are given by (3.26*b*), (3.30*c*) and (3.51*c*) respectively. Next, we add (4.3*a*) and (4.4) to eliminate $\boldsymbol{v}^L$:

$$\boldsymbol{v}^S + \boldsymbol{V}^P = \boldsymbol{V} - \boldsymbol{v} = \left(\boldsymbol{v}^s + \boldsymbol{V}^p\right) + \left(\boldsymbol{v}^\dagger + \boldsymbol{V}^\dagger\right). \tag{4.5a}$$

Then, recalling that $\boldsymbol{V}^E - \boldsymbol{v} = \boldsymbol{v}^s + \boldsymbol{V}^p$ (see (3.51a)), (4.5a) yields the result

$$\boldsymbol{V} - \boldsymbol{V}^E = \boldsymbol{v}^\dagger + \boldsymbol{V}^\dagger, \tag{4.5b}$$

which we need in the next section to understand the relation between the equations for $\boldsymbol{V}$ and $\boldsymbol{V}^E$. Essentially, the sum $\boldsymbol{v}^\dagger + \boldsymbol{V}^\dagger$ measures the difference $\boldsymbol{V} - \boldsymbol{V}^E$.

Now we may average our various forms of Euler's equation to obtain several mean Euler equations. First, the average of the elementary representation $\boldsymbol{\mathscr{E}} = \boldsymbol{0}$ with $\boldsymbol{\mathscr{E}}$ given by (2.23b) is

$$\overline{\boldsymbol{\mathscr{E}}} = \partial_t \overline{\boldsymbol{v}} + \{\overline{\boldsymbol{v}}, \overline{\boldsymbol{v}}\} + \overline{\{\boldsymbol{v}', \boldsymbol{v}'\}} + \nabla \overline{\Pi} = \boldsymbol{0}, \qquad \nabla\cdot\overline{\boldsymbol{v}} = 0. \tag{4.6a,b}$$

In consequence, for material closed curves $\mathscr{C}^E$ moving with the mean velocity $\overline{\boldsymbol{v}}$, Kelvin's circulation theorem (2.29b) takes the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\oint_{\mathscr{C}^E} \overline{\boldsymbol{v}}\cdot\mathrm{d}\boldsymbol{x} = -\oint_{\mathscr{C}^E} \overline{\{\boldsymbol{v}', \boldsymbol{v}'\}}\cdot\mathrm{d}\boldsymbol{x}. \tag{4.6c}$$

Second, the average of (3.48b) is

$$\overline{\boldsymbol{E}^E} = \partial_t \overline{\boldsymbol{V}^E} + \left\{\overline{\boldsymbol{v}}, \overline{\boldsymbol{V}^E}\right\} + \overline{\{\boldsymbol{\zeta}, \{\boldsymbol{v}', \boldsymbol{v}\}\}} + \nabla\overline{\Pi^E} = \boldsymbol{0}, \tag{4.7a}$$

for which

$$\frac{\mathrm{d}}{\mathrm{d}t}\oint_{\mathscr{C}^E} \overline{\boldsymbol{V}^E}\cdot\mathrm{d}\boldsymbol{x} = -\oint_{\mathscr{C}^E} \overline{\{\boldsymbol{\zeta}, \{\boldsymbol{v}', \boldsymbol{v}\}\}}\cdot\mathrm{d}\boldsymbol{x} \tag{4.7b}$$

or alternatively, on use of the identity (A2d) in Appendix A in the definition (3.45a) of $\boldsymbol{V}^E$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\oint_{\mathscr{C}^E} (\overline{\boldsymbol{v}} - \overline{\boldsymbol{\zeta}\times(\nabla\times\boldsymbol{v}')})\cdot\mathrm{d}\boldsymbol{x} = -\oint_{\mathscr{C}^E} \overline{\{\boldsymbol{\zeta}, \{\boldsymbol{v}', \boldsymbol{v}\}\}}\cdot\mathrm{d}\boldsymbol{x} \tag{4.7c}$$

(cf. eq. (4.11) on p. 275 of Holm (2002b); there, of course, the right-hand side of (4.7c) is replaced by zero, but otherwise his eq. (4.11) is equivalent).

As in all mean-field approaches, the heart of the difficulty in the use of (4.6c) and (4.7b) lies in the evaluation of the averages $\overline{\{\boldsymbol{v}', \boldsymbol{v}'\}}$ and $\overline{\{\boldsymbol{\zeta}, \{\boldsymbol{v}', \boldsymbol{v}\}\}}$ respectively. To obtain them we need knowledge of $\boldsymbol{v}'$ and, in the case of (4.7), of $\boldsymbol{\zeta}$ also.

In the case of the HEL formulations we may proceed similarly and average (3.52a) to obtain

$$\overline{\boldsymbol{E}} = \partial_t \overline{\boldsymbol{V}} + \left\{\overline{\boldsymbol{v}}, \overline{\boldsymbol{V}}\right\} + \left\{\overline{\boldsymbol{v}^S}, \overline{\boldsymbol{V}}\right\} + \nabla\overline{\Pi^L} = \boldsymbol{0} \tag{4.8a}$$

so that the average of (3.53) becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}\oint_{\mathscr{C}^E} \overline{\boldsymbol{V}}\cdot\mathrm{d}\boldsymbol{x} = -\oint_{\mathscr{C}^E} \left\{\overline{\boldsymbol{v}^S}, \overline{\boldsymbol{V}}\right\}\mathrm{d}\boldsymbol{x}. \tag{4.8b}$$

Finally the average of (3.28) with $\boldsymbol{E}$ given by (3.27b) is

$$\overline{\boldsymbol{E}} = \partial_t \overline{\boldsymbol{V}} + \left\{\overline{\boldsymbol{v}^L}, \overline{\boldsymbol{V}}\right\} + \nabla\overline{\Pi^L} = \boldsymbol{0} \tag{4.9a}$$

so that the average of (3.31b) is simply

$$\frac{\mathrm{d}}{\mathrm{d}t}\oint_{\mathscr{C}^L} \overline{\boldsymbol{V}}\cdot\mathrm{d}\boldsymbol{x} = \boldsymbol{0}. \tag{4.9b}$$

The point that we wish to emphasize is that, in view of (4.6c), (4.7b) and (4.8b), none of the circuit integrals $\oint_{\mathscr{C}^E} \overline{\boldsymbol{v}}\cdot\mathrm{d}\boldsymbol{x}$, $\oint_{\mathscr{C}^E} \overline{\boldsymbol{V}^E}\cdot\mathrm{d}\boldsymbol{x}$ and $\oint_{\mathscr{C}^E} \overline{\boldsymbol{V}}\cdot\mathrm{d}\boldsymbol{x}$ remain constant when advected by the mean flow $\overline{\boldsymbol{v}}$. On the other hand, (4.9b) shows that $\oint_{\mathscr{C}^L} \overline{\boldsymbol{V}}\cdot\mathrm{d}\boldsymbol{x}$ remains constant when advected by the mean flow $\overline{\boldsymbol{v}^L}$. These results illustrate the fact that any mean field theory that wishes to preserve mean circulation under advection by a mean flow must choose that flow to be $\overline{\boldsymbol{v}^L}$.

5. Small-amplitude fluctuations

The aim of this section is to clarify the relationship between the Eulerian and Lagrangian approaches of §3. The main difficulty in reaching this objective arises from the nature of the Lagrangian approach, which describes properties at a displaced point $\boldsymbol{x}^L$ rather than at $\boldsymbol{x}$ itself. Fortunately the key facts emerge from an analysis that assumes the displacements are small and that evaluates quantities only to second order in $\zeta \equiv |\boldsymbol{\zeta}|$ and $\xi \equiv |\boldsymbol{\xi}|$. It is seen below that $\boldsymbol{\zeta}$ and $\boldsymbol{\xi}$ differ only at second order, so that $\boldsymbol{\xi}$ and $\boldsymbol{\zeta}$ may be used interchangeably in second-order terms and $O(\xi^3)$ errors can be expressed as $O(\zeta^3)$ errors. An example of this is the Taylor expansion of the Lagrangian scalar $\phi^L(\boldsymbol{x}, t) = \phi(\boldsymbol{x}+\boldsymbol{\xi}, t)$ (see (3.4*b*) and (3.5*a*)), which in terms of $\phi(\boldsymbol{x})$ is

$$\phi^L = \phi + \boldsymbol{\xi}\boldsymbol{\cdot}\nabla\phi + \tfrac{1}{2}\xi_j\xi_k\frac{\partial^2\phi}{\partial x_j \partial x_k} + O(\xi^3). \tag{5.1}$$

We will eventually replace the second-order product $\xi_j\xi_k$ in (5.1) by $\zeta_j\zeta_k$ and will write the error term as $O(\zeta^3)$, but we will be careful to distinguish between a first-order term such as $\boldsymbol{\xi}\boldsymbol{\cdot}\nabla$ and its ζ-counterpart $\boldsymbol{\zeta}\boldsymbol{\cdot}\nabla$.

Use of (5.1) in the case of the velocity shows that $\boldsymbol{v}^S$ defined by (4.3*a*) has the Taylor expansion

$$\boldsymbol{v}^S \equiv \boldsymbol{v}^L - \boldsymbol{v} = \boldsymbol{\xi}\boldsymbol{\cdot}\nabla\boldsymbol{v} + \tfrac{1}{2}\xi_j\xi_k\frac{\partial^2\boldsymbol{v}}{\partial x_j \partial x_k} + O(\xi^3). \tag{5.2}$$

Its mean and fluctuating parts determine

$$\overline{\boldsymbol{v}^L} = \overline{\boldsymbol{v}} + O(\xi^2), \qquad \boldsymbol{v}^{S'} = \boldsymbol{\xi}\boldsymbol{\cdot}\nabla\overline{\boldsymbol{v}} + O(\xi^2). \tag{5.3a,b}$$

Then from (5.3*b*) and (4.3*b*), we obtain

$$\boldsymbol{\zeta} = \boldsymbol{\xi} + O(\xi^2) \qquad \text{so that} \qquad \boldsymbol{v}^{S'} = \boldsymbol{v}^{s'} + O(\xi^2) \tag{5.3c,d}$$

(see (3.51*b*)). Since trivially (5.3*c*) implies that $\boldsymbol{\xi} = \boldsymbol{\zeta} + O(\zeta^2)$ and, as we find it more convenient to work with $\boldsymbol{\zeta}$ rather than $\boldsymbol{\xi}$, we henceforth give all error estimates in terms of ζ.

The value of the fluctuating vector $\boldsymbol{v}^{S'} - \boldsymbol{\xi}\boldsymbol{\cdot}\nabla\overline{\boldsymbol{v}^L}$ is given exactly by (4.3*b*) but is also determined approximately by (5.2). On equating the two expressions, we determine the evolution equation

$$\partial_t(\boldsymbol{\xi}-\boldsymbol{\zeta}) + [\overline{\boldsymbol{v}}, (\boldsymbol{\xi}-\boldsymbol{\zeta})] \approx \left(\boldsymbol{\zeta}\boldsymbol{\cdot}\nabla\boldsymbol{v}'\right)' + \tfrac{1}{2}(\zeta_j\zeta_k)'\frac{\partial^2\overline{\boldsymbol{v}}}{\partial x_j \partial x_k} = O(\zeta^2) \tag{5.4}$$

for the difference $\boldsymbol{\xi}-\boldsymbol{\zeta}$, where here and henceforth equality correct to order ζ^2 is denoted by $\approx$. Fortunately, when we consider mean quantities, we only need $O(\zeta)$ accuracy for $\boldsymbol{\xi}-\boldsymbol{\zeta}$ and so we never need to solve (5.4) to determine its smaller $O(\zeta^2)$ part.

Since the traditional GLM-approach developed in §3.1 within the HEL framework concerns the evolution of $\overline{\boldsymbol{V}} = \overline{\boldsymbol{v}} + \overline{\boldsymbol{v}^S} + \overline{\boldsymbol{V}^P}$, while Holm's glm-approach, as developed in §3.2, concerns the evolution of $\overline{\boldsymbol{V}^E} = \overline{\boldsymbol{v}} + \overline{\boldsymbol{v}^s} + \overline{\boldsymbol{V}^p}$, we will now identify the values of the constituent velocities as well as determine the difference $\overline{\boldsymbol{V}} - \overline{\boldsymbol{V}^E} = \overline{\boldsymbol{v}^\dagger} + \overline{\boldsymbol{V}^\dagger}$ (see (4.5*b*)) in the small-ζ limit (see (5.5*d*), (5.6*d*) below).

Subtraction of $\boldsymbol{v}^s = \boldsymbol{\zeta}\boldsymbol{\cdot}\nabla\boldsymbol{v}$ (see (3.51*b*)) from $\boldsymbol{v}^S$ given by (5.2) determines $\boldsymbol{v}^\dagger$ defined by (4.3*a*). Correct to our required $O(\zeta^2)$ accuracy it is

$$\boldsymbol{v}^\dagger \equiv \boldsymbol{v}^S - \boldsymbol{v}^s \approx (\boldsymbol{\xi}-\boldsymbol{\zeta})\boldsymbol{\cdot}\nabla\overline{\boldsymbol{v}} + \tfrac{1}{2}\zeta_j\zeta_k\frac{\partial^2\overline{\boldsymbol{v}}}{\partial x_j\partial x_k}, \tag{5.5a}$$

from which we deduce that

$$\overline{\boldsymbol{v}^S} = \overline{\boldsymbol{v}^s} + \overline{\boldsymbol{v}^\dagger}, \tag{5.5b}$$

where

$$\overline{\boldsymbol{v}^s} = \overline{\boldsymbol{\zeta}\boldsymbol{\cdot}\nabla\boldsymbol{v}'} \qquad \text{and} \qquad \overline{\boldsymbol{v}^\dagger} \approx \tfrac{1}{2}\overline{\zeta_j\zeta_k}\frac{\partial^2\overline{\boldsymbol{v}}}{\partial x_j\partial x_k}. \tag{5.5c, d}$$

Unfortunately the fluctuating part $\boldsymbol{v}^{\dagger\prime}$ of $\boldsymbol{v}^\dagger$ depends on $\boldsymbol{\xi}-\boldsymbol{\zeta}$ for which no closed-form $O(\zeta^2)$ solution of (5.4) is available. Nevertheless, this is of no consequence, as we only need $O(\zeta)$ accuracy of our fluctuating quantities.

In a similar vein, subtraction of $\boldsymbol{V}^p = (\nabla\boldsymbol{\zeta})\boldsymbol{\cdot}\boldsymbol{v}$ (see (3.51*c*)) from $\boldsymbol{V}^P = (\nabla\boldsymbol{\xi})\boldsymbol{\cdot}\boldsymbol{v}^L$ given by (3.30*c*) determines $\boldsymbol{V}^\dagger$ defined by (4.4). Remembering that $\boldsymbol{v} = \boldsymbol{v}^L - \boldsymbol{v}^S$, we obtain

$$\boldsymbol{V}^\dagger \equiv \boldsymbol{V}^P - \boldsymbol{V}^p = \big(\nabla(\boldsymbol{\xi}-\boldsymbol{\zeta})\big)\boldsymbol{\cdot}\boldsymbol{v}^L + (\nabla\boldsymbol{\zeta})\boldsymbol{\cdot}\boldsymbol{v}^S \tag{5.6a}$$

from which we see that

$$\overline{\boldsymbol{V}^P} = \overline{\boldsymbol{V}^p} + \overline{\boldsymbol{V}^\dagger}, \tag{5.6b}$$

where

$$\overline{\boldsymbol{V}^p} = \overline{(\nabla\boldsymbol{\zeta})\boldsymbol{\cdot}\boldsymbol{v}'} \qquad \text{and} \qquad \overline{V_i^\dagger} \approx \overline{\zeta_k\frac{\partial\zeta_j}{\partial x_i}}\frac{\partial\overline{v}_j}{\partial x_k} \tag{5.6c, d}$$

on use of $\boldsymbol{v}^S = \boldsymbol{\zeta}\boldsymbol{\cdot}\nabla\overline{\boldsymbol{v}} + O(\zeta^2)$ (see (5.2)).

The mean Euler equations $\overline{\boldsymbol{E}} = \boldsymbol{0}$ (see (4.8*a*)) and $\overline{\boldsymbol{E}^E} = \boldsymbol{0}$ (see (4.7*a*)) are not only concerned with advection of momentum identified by $\overline{\boldsymbol{V}}$ and $\overline{\boldsymbol{V}^E}$ but also involve the gradients of the pressures $\overline{\Pi^L}$ and $\overline{\Pi^E}$ respectively. To compare these pressure gradient terms, we take the Taylor expansion (5.1) for Π^L, subtract from it the value $\Pi^E = \Pi + \boldsymbol{\zeta}\boldsymbol{\cdot}\nabla\Pi$ given by (3.45*b*) and so obtain

$$\Pi^\dagger \equiv \Pi^L - \Pi^E \approx (\boldsymbol{\xi}-\boldsymbol{\zeta})\boldsymbol{\cdot}\nabla\Pi + \tfrac{1}{2}\zeta_j\zeta_k\frac{\partial^2\Pi}{\partial x_j\partial x_k}, \tag{5.7a}$$

from which we deduce that

$$\overline{\Pi^L} = \overline{\Pi^E} + \overline{\Pi^\dagger}, \tag{5.7b}$$

where

$$\overline{\Pi^E} = \overline{\Pi} + \overline{\boldsymbol{\zeta}\boldsymbol{\cdot}\nabla\Pi'} \qquad \text{and} \qquad \overline{\Pi^\dagger} \approx \tfrac{1}{2}\overline{\zeta_j\zeta_k}\frac{\partial^2\overline{\Pi}}{\partial x_j\partial x_k}. \tag{5.7c, d}$$

Finally to complete the links between the Euler equations themselves, we present the decompositions suggested by our treatment of the velocity above. As in the representation (5.5*a*) of $\boldsymbol{v}^\dagger$, we introduce

$$\boldsymbol{\mathscr{E}}^\dagger \equiv \boldsymbol{\mathscr{E}}^L - (\boldsymbol{\mathscr{E}} + \boldsymbol{\zeta}\boldsymbol{\cdot}\nabla\boldsymbol{\mathscr{E}}) \approx (\boldsymbol{\xi}-\boldsymbol{\zeta})\boldsymbol{\cdot}\nabla\boldsymbol{\mathscr{E}} + \tfrac{1}{2}\zeta_j\zeta_k\frac{\partial^2\boldsymbol{\mathscr{E}}}{\partial x_j\partial x_k}, \tag{5.8a}$$

and, in a similar spirit to the representation (5.6*a*) of $\boldsymbol{V}^\dagger$, we introduce

$$\boldsymbol{E}^\dagger \equiv (\boldsymbol{E} - \boldsymbol{\mathscr{E}}^L) - (\nabla\boldsymbol{\zeta})\boldsymbol{\cdot}\boldsymbol{\mathscr{E}} = \big(\nabla(\boldsymbol{\xi}-\boldsymbol{\zeta})\big)\boldsymbol{\cdot}\boldsymbol{\mathscr{E}}^L + (\nabla\boldsymbol{\zeta})\boldsymbol{\cdot}(\boldsymbol{\mathscr{E}}^L - \boldsymbol{\mathscr{E}}), \tag{5.8b}$$

where we have used $\boldsymbol{E}-\boldsymbol{\mathscr{E}}^L = (\nabla\boldsymbol{\xi})\boldsymbol{\cdot}\boldsymbol{\mathscr{E}}^L$ (see (3.27*c*)). Moreover, since $\boldsymbol{E}^E = \boldsymbol{\mathscr{E}}+\{\boldsymbol{\zeta}\,,\,\boldsymbol{\mathscr{E}}\}$ (see (3.48*a*)), we may write the first equality in (5.8*a*) in the alternative form $\boldsymbol{\mathscr{E}}^\dagger = (\boldsymbol{\mathscr{E}}^L - \boldsymbol{E}^E) + (\nabla\boldsymbol{\zeta})\boldsymbol{\cdot}\boldsymbol{\mathscr{E}}$, which when added to (5.8*b*) yields

$$\boldsymbol{E} - \boldsymbol{E}^E = \boldsymbol{\mathscr{E}}^\dagger + \boldsymbol{E}^\dagger \tag{5.8c}$$

(cf. (4.5*b*)). Thus the averaged forms of Euler's equation are related by

$$\overline{\boldsymbol{E}} - \overline{\boldsymbol{E}^E} = \overline{\boldsymbol{\mathscr{E}}^\dagger} + \overline{\boldsymbol{E}^\dagger}\,, \tag{5.9a}$$

where

$$\overline{\boldsymbol{\mathscr{E}}^\dagger} \approx \tfrac{1}{2}\overline{\zeta_j\zeta_k}\frac{\partial^2\overline{\boldsymbol{\mathscr{E}}}}{\partial x_j\partial x_k} \qquad \text{and} \qquad \overline{E_i^\dagger} \approx \overline{\zeta_k\frac{\partial\zeta_j}{\partial x_i}}\frac{\partial\overline{\mathscr{E}_j}}{\partial x_k} \tag{5.9b, c}$$

on use of the result $\boldsymbol{\mathscr{E}}^L - \boldsymbol{\mathscr{E}} = \boldsymbol{\zeta}\boldsymbol{\cdot}\nabla\boldsymbol{\mathscr{E}} + O(\zeta^2)$ obtained from (5.8*a*) (cf. the expressions (5.5*d*) and (5.6*d*) for $\overline{\boldsymbol{v}^\dagger}$ and $\overline{\boldsymbol{V}^\dagger}$ respectively).

The difference of $\overline{\boldsymbol{E}}$ and $\overline{\boldsymbol{E}^E}$ determined by (5.9*a*) explains the relevance of $\overline{\boldsymbol{\mathscr{E}}^\dagger}$ and $\overline{\boldsymbol{E}^\dagger}$. That difference exposes clearly the way in which the structure of the glm-equation $\overline{\boldsymbol{E}^E} = \boldsymbol{0}$ (as corrected by us) differs from that of the GLM-equation $\overline{\boldsymbol{E}} = \boldsymbol{0}$.

6. Conclusions

In this paper we have derived consequences of Hamilton's principle for the motion of an incompressible fluid. All our results generalize easily to compressible fluids, for which the pressure term $-p((\rho/\rho_0)-1)$ in (2.9*a*) is replaced by $\rho U(\rho, S)$ where U is the internal energy per unit mass and S is the specific entropy. In the special case of isentropy, the compressible fluid is barotropic ($U = U(\rho)$) and Kelvin's theorem again applies together with all our principal results including the one that prevents the NS–α equations from obeying Kelvin's circulation theorem.

Our main conclusion concerns the use of Hamilton's principle when the flow velocity $\boldsymbol{v}$ is decomposed into two parts $\overline{\boldsymbol{v}}$ and $\boldsymbol{v}'$. Recall that in our §3.2 development we regarded this decomposition as arbitrary and not constrained by our later demand that $\overline{\boldsymbol{v}}$ and $\boldsymbol{v}'$ are the mean and fluctuating parts of $\boldsymbol{v}$. Without that constraint, any velocity increment $\delta\boldsymbol{v}$ resulting from a virtual displacement of a particle path may be expressed as $\delta\boldsymbol{v} = \delta\overline{\boldsymbol{v}} + \delta\boldsymbol{v}'$, in which the choice of the values of one or other of $\delta\overline{\boldsymbol{v}}$ and $\delta\boldsymbol{v}'$ is at our disposal. So, if we make the traditional choice $\delta\boldsymbol{v} = \delta\overline{\boldsymbol{v}}$ corresponding to fixed $\boldsymbol{v}'$ with $\delta\boldsymbol{v}' = \boldsymbol{0}$, we obviously recover the variational form (3.44) of Euler's equation without the extra term (3.46) involving $\delta A^E/\delta\boldsymbol{\zeta}$. We say 'obviously', because the derivation is equivalent to the original §2.2 derivation of Euler's equation $\boldsymbol{\mathscr{E}} = \boldsymbol{0}$ with $\boldsymbol{\mathscr{E}}$ given by (2.26*a*). In §3.2, however, we adopt the representation $\boldsymbol{v}' = \partial_t\boldsymbol{\zeta} + [\overline{\boldsymbol{v}}\,,\,\boldsymbol{\zeta}]$, which means that not only is $\boldsymbol{v}'$ a function of a new vector $\boldsymbol{\zeta}$ it is also a function of $\overline{\boldsymbol{v}}$. Consequently, when $\boldsymbol{\zeta}$ is held fixed, the increment $\delta\overline{\boldsymbol{v}}$ forces the simultaneous increment $\delta\boldsymbol{v}' = [\delta\overline{\boldsymbol{v}}\,,\,\boldsymbol{\zeta}]$ so that $\delta\boldsymbol{v} \neq \delta\overline{\boldsymbol{v}}$. To accommodate this failure we have to include a variation of $\boldsymbol{\zeta}$ at fixed $\overline{\boldsymbol{v}}$ to achieve the correct value of $\delta\boldsymbol{v}$. The consequences are encapsulated in the variational form (3.44) of Euler's equation which includes the term involving $\delta A^E/\delta\boldsymbol{\zeta}$. From an operational point of view all the results just described continue to apply when $\overline{\boldsymbol{v}}$ and $\boldsymbol{v}'$ are the mean and fluctuating parts of $\boldsymbol{v}$.

In essence, Holm (2002*b*) adopted the proposed representation $\boldsymbol{v}' = \partial_t\boldsymbol{\zeta} + [\overline{\boldsymbol{v}}\,,\,\boldsymbol{\zeta}]$ but mistakenly used the variational equations that arise from the traditional choice $\delta\boldsymbol{v} = \delta\overline{\boldsymbol{v}}$. Having ignored the consequences of the necessary displacement of $\boldsymbol{\zeta}$, he arrived at (4.7*a*) without the term $\overline{\{\boldsymbol{\zeta}\,,\,\{\boldsymbol{v}'\,,\,\boldsymbol{v}\}\}} = -\overline{\boldsymbol{\zeta}\times(\nabla\times\{\boldsymbol{v}'\,,\,\boldsymbol{v}\})} + \nabla\overline{\boldsymbol{\zeta}\boldsymbol{\cdot}\{\boldsymbol{v}'\,,\,\boldsymbol{v}\}}$, as determined by the alternative representation (A2*d*) in Appendix A of the bilinear operator $\{\cdot\,,\,\cdot\}$. This attractive omission has the consequence that the circulation $\oint_{\mathscr{C}^E}\overline{\boldsymbol{V}^E}\boldsymbol{\cdot}\,\mathrm{d}\boldsymbol{x}$ about contours $\mathscr{C}^E$ advected at the mean velocity $\overline{\boldsymbol{v}}$ is conserved. If conservation of circulation is of the first importance, it can also be achieved by making the $\overline{\text{integral on the right-hand side of (4.7}b\text{) vanish through}}$ the closure assumption, $\boldsymbol{\zeta}\times(\nabla\times\{\boldsymbol{v}',\,\boldsymbol{v}\})=\nabla\overline{\phi^E}$ for some single valued function $\overline{\phi^E}$. A simpler alternative way of conserving circulation $\oint_{\mathscr{C}^E}\overline{\boldsymbol{v}}\boldsymbol{\cdot}\,\mathrm{d}\boldsymbol{x}$ is to make the closure assumption $\overline{\boldsymbol{v}'\times(\nabla\times\boldsymbol{v}')}=\nabla\overline{\phi}$ in (4.6*a*), for some single-valued function $\overline{\phi}$. Of course this assumption encompasses the simplistic classical approximation, which neglects the Reynolds stress $\overline{\boldsymbol{v}'\boldsymbol{\cdot}\nabla\boldsymbol{v}'}$ in its entirety. There is a third alternative suggested by (4.8*a*): if we make the assumption $\overline{\boldsymbol{v}^S}\times(\nabla\times\overline{\boldsymbol{V}})=\nabla\overline{\phi^L}$ for some single-valued function $\overline{\phi^L}$, then the circulation $\oint_{\mathscr{C}^E}\overline{\boldsymbol{V}}\boldsymbol{\cdot}\,\mathrm{d}\boldsymbol{x}$ is preserved. It is clear that the adoption of any of these assumptions, including Holm's implicit assumption that $\overline{\{\boldsymbol{\zeta}\,,\{\boldsymbol{v}'\,,\,\boldsymbol{v}\}\}}=\boldsymbol{0}$, requires a physical argument in support. In Holm's case, since researchers appear to be unaware of the issue, no argument for the neglect of $\overline{\{\boldsymbol{\zeta}\,,\{\boldsymbol{v}'\,,\,\boldsymbol{v}\}\}}$ has ever been given.

The upshot of all this is that the proper way to achieve conservation of circulation is to adopt the HEL approach which yields the mean field equation (4.9*a*). That, in turn, leads to the conservation of the circulation $\oint_{\mathscr{C}^L}\overline{\boldsymbol{V}}\boldsymbol{\cdot}\,\mathrm{d}\boldsymbol{x}$ (see (4.9*b*)) for circuits advected by the mean Lagrangian velocity $\overline{\boldsymbol{v}^L}=\overline{\boldsymbol{v}}+\overline{\boldsymbol{v}^S}$ rather than the Eulerian mean $\overline{\boldsymbol{v}}$. Evidently the study of the evolution of $\overline{\boldsymbol{V}}$ is relevant. An investigation along the lines of Holm (2002*b*) must be based on the small-ξ assumption made in §5. Holm (2002*b*) made effective use of various Taylor hypotheses (closure approximations) and in particular of $\overline{\boldsymbol{V}^E}-\overline{\boldsymbol{v}}=\overline{\boldsymbol{v}^s}+\overline{\boldsymbol{V}^p}\approx-\alpha^2\nabla^2\overline{\boldsymbol{v}}$, for some α determined by the quadratic moments of the displacement statistics. He could do this because of the relatively simple forms (5.5*c*) and (5.6*c*) of $\overline{\boldsymbol{v}^s}$ and $\overline{\boldsymbol{V}^p}$. It is far from clear that any comparably simple α representation is available for the additional difference $\overline{\boldsymbol{V}}-\overline{\boldsymbol{V}^E}=\overline{\boldsymbol{v}^\dagger}+\overline{\boldsymbol{V}^\dagger}$ defined by (4.5*b*) and determined by the approximate (and more complicated) representations (5.5*d*) and (5.6*d*) of $\overline{\boldsymbol{v}^\dagger}$ and $\overline{\boldsymbol{V}^\dagger}$. Whatever mean-field approach is adopted, it should be remembered that only $\overline{\boldsymbol{v}}$ has the solenoidal property $\nabla\boldsymbol{\cdot}\overline{\boldsymbol{v}}=0$, while other vectors such as $\overline{\boldsymbol{v}^S}$, $\overline{\boldsymbol{v}^L}$, $\overline{\boldsymbol{V}^E}$ and $\overline{\boldsymbol{V}}$ are generally not solenoidal.

Finally it should be pointed out that the principal results of this paper can be derived by attacking (2.1*a*) directly, without invoking Hamilton's principle, which is in any case unavailable when viscosity is restored. Moreover, the same direct approach is successful when viscosity is included, and the Navier–Stokes equations are attacked. Under the HEL transformation, the viscous term $\nu\nabla^2\boldsymbol{v}$ becomes complicated; see Roberts & Soward (2006). We shall therefore not consider this generalization here.

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Appendix A. The bilinear operators

Note that, though the vectors used in the identities (A1)–(A3) below are arbitrary, they have been chosen to reflect our usage in this paper.

The bilinear operator

$$[\boldsymbol{v}\,,\,\boldsymbol{\eta}] \equiv \boldsymbol{v}\boldsymbol{\cdot}\nabla\boldsymbol{\eta} - \boldsymbol{\eta}\boldsymbol{\cdot}\nabla\boldsymbol{v} \tag{A1a}$$

introduced in (2.21c) is antisymmetric, $[\boldsymbol{\eta}\,,\,\boldsymbol{v}] + [\boldsymbol{v}\,,\,\boldsymbol{\eta}] = \boldsymbol{0}$. It has the divergence property

$$\nabla\boldsymbol{\cdot}\big(\phi\,[\boldsymbol{v}\,,\,\boldsymbol{\eta}]\big) = \nabla\boldsymbol{\cdot}\big(\boldsymbol{v}\,\nabla\boldsymbol{\cdot}(\phi\,\boldsymbol{\eta}) - \boldsymbol{\eta}\,\nabla\boldsymbol{\cdot}(\phi\,\boldsymbol{v})\big) \tag{A1b}$$

and the useful cyclic property

$$[\,[\overline{\boldsymbol{v}}\,,\,\boldsymbol{\zeta}]\,,\,\boldsymbol{\eta}] + [\,[\boldsymbol{\eta}\,,\,\overline{\boldsymbol{v}}]\,,\,\boldsymbol{\zeta}] + [\,[\boldsymbol{\zeta}\,,\,\boldsymbol{\eta}]\,,\,\overline{\boldsymbol{v}}] = \boldsymbol{0} \tag{A1c}$$

called the Jacobi identity in Lie algebra.

The bilinear operators introduced in (2.22b) and (2.25b) have the component forms

$$\{\boldsymbol{v}\,,\,\boldsymbol{V}\}_i = v_k\frac{\partial V_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i}V_k\,, \qquad \langle\boldsymbol{v}\,,\,\boldsymbol{V}\rangle_i = \frac{\partial}{\partial x_k}(v_k V_i) + \frac{\partial v_k}{\partial x_i}V_k\,. \tag{A2a,b}$$

They are related by

$$\{\boldsymbol{v}\,,\,\boldsymbol{V}\} \equiv \langle\boldsymbol{v}\,,\,\boldsymbol{V}\rangle - (\nabla\boldsymbol{\cdot}\boldsymbol{v})\,\boldsymbol{V}\,, \tag{A2c}$$

a result which shows that $\{\boldsymbol{v}\,,\,\boldsymbol{V}\}$ and $\langle\boldsymbol{v}\,,\,\boldsymbol{V}\rangle$ are the same whenever $\nabla\boldsymbol{\cdot}\boldsymbol{v} = 0$. In the concluding §6, we appeal to the alternative form

$$\{\boldsymbol{v}\,,\,\boldsymbol{V}\} \equiv -\boldsymbol{v}\times(\nabla\times\boldsymbol{V}) + \nabla(\boldsymbol{v}\boldsymbol{\cdot}\boldsymbol{V})\,. \tag{A2d}$$

In our application of the bilinear operator (A1b) to Hamilton's principle, particularly in §3.2, we need the adjoint property

$$\boldsymbol{V}\boldsymbol{\cdot}[\boldsymbol{v}\,,\,\boldsymbol{\eta}] + \boldsymbol{\eta}\boldsymbol{\cdot}\langle\boldsymbol{v}\,,\,\boldsymbol{V}\rangle = \nabla\boldsymbol{\cdot}((\boldsymbol{\eta}\boldsymbol{\cdot}\boldsymbol{V})\,\boldsymbol{v}) \tag{A3a}$$

and the useful commutator property

$$\langle\overline{\boldsymbol{v}}\,,\langle\boldsymbol{\zeta}\,,\,\boldsymbol{V}\rangle\rangle - \langle\boldsymbol{\zeta}\,,\langle\overline{\boldsymbol{v}}\,,\,\boldsymbol{V}\rangle\rangle = \langle[\overline{\boldsymbol{v}},\boldsymbol{\zeta}],\boldsymbol{V}\rangle, \tag{A3b}$$

which is related to the result (A1c).

Appendix B. Verification of the key formula (3.43) for $\mathscr{E}$

We consider the first three terms on the right-hand side of (3.37a). As instructed in §3.2, we substitute into each successive term the values of $\delta\overline{\boldsymbol{v}}$, $\delta\overline{\rho}$ and $\delta\boldsymbol{\zeta}$ given by (3.41a, b) and (3.42) respectively. Then use of the adjoint property (A3a) gives

$$\begin{aligned}&\left(\frac{\delta A^E}{\delta\overline{\boldsymbol{v}}}\right)\boldsymbol{\cdot}\delta\overline{\boldsymbol{v}} + \left(\frac{\delta A^E}{\delta\overline{\rho}}\right)\delta\overline{\rho} + \left(\frac{\delta A^E}{\delta\boldsymbol{\zeta}}\right)\boldsymbol{\cdot}\delta\boldsymbol{\zeta} + \boldsymbol{\eta}\boldsymbol{\cdot}\rho\mathscr{E}\\ &\quad = \frac{\partial}{\partial t}\left(\boldsymbol{\eta}\boldsymbol{\cdot}\frac{\delta A^E}{\delta\overline{\boldsymbol{v}}}\right) + \nabla\boldsymbol{\cdot}\left(\left(\boldsymbol{\eta}\boldsymbol{\cdot}\frac{\delta A^E}{\delta\overline{\boldsymbol{v}}}\right)\overline{\boldsymbol{v}} - \boldsymbol{\eta}\overline{\rho}\frac{\delta A^E}{\delta\overline{\rho}} + \left(\boldsymbol{\eta}\boldsymbol{\cdot}\frac{\delta A^E}{\delta\boldsymbol{\zeta}}\right)\boldsymbol{\zeta}\right),\end{aligned} \tag{B1}$$

in which $\rho\mathscr{E}$ is defined by (3.43). On the right-hand side of (B1) we substitute the values (3.37$b-d$) for $\delta A^E/\delta\overline{\boldsymbol{v}}$, $\delta A^E/\delta\boldsymbol{\zeta}$ and $\delta A^E/\delta\overline{\rho}$. To the resulting expression, we now add the remaining terms in (3.37a), again using the expressions (3.41a, b) and (3.42) for $\delta\overline{\boldsymbol{v}}$, $\delta\overline{\rho}$ and $\delta\boldsymbol{\zeta}$. Accordingly, we obtain the result

$$\delta L + \boldsymbol{\eta}\boldsymbol{\cdot}\rho\mathscr{E} = \partial_t\big(\boldsymbol{\eta}\boldsymbol{\cdot}\rho\boldsymbol{v} + \boldsymbol{M}\big) + \nabla\boldsymbol{\cdot}\big((\boldsymbol{\eta}\boldsymbol{\cdot}\rho\boldsymbol{v})(\overline{\boldsymbol{v}} + \boldsymbol{v}') + \boldsymbol{\eta}(\overline{\rho} + \rho')\Pi + \boldsymbol{S}^v + \mathbf{S}^\Pi\big), \tag{B2}$$

in which $\boldsymbol{v}'$, ρ' are given by (3.32c, d) and

$$\boldsymbol{M} \,=\, \rho\boldsymbol{v}\boldsymbol{\cdot}[\boldsymbol{\zeta}\,,\,\boldsymbol{\eta}] + \boldsymbol{\eta}\boldsymbol{\cdot}\langle\boldsymbol{\zeta}\,,\,\rho\boldsymbol{v}\rangle - \nabla\boldsymbol{\cdot}((\rho\boldsymbol{v}\boldsymbol{\cdot}\boldsymbol{\eta})\,\boldsymbol{\zeta}), \tag{B3a}$$

$$\mathbf{S}^{v} \,=\, \overline{\boldsymbol{v}}\big(\rho\boldsymbol{v}\boldsymbol{\cdot}[\boldsymbol{\zeta}\,,\,\boldsymbol{\eta}] + \boldsymbol{\eta}\boldsymbol{\cdot}\langle\boldsymbol{\zeta}\,,\,\rho\boldsymbol{v}\rangle\big) - \boldsymbol{\zeta}\big(\rho\boldsymbol{v}\boldsymbol{\cdot}[\overline{\boldsymbol{v}}\,,\,\boldsymbol{\eta}] + \boldsymbol{\eta}\boldsymbol{\cdot}\langle\overline{\boldsymbol{v}}\,,\,\rho\boldsymbol{v}\rangle\big) - (\boldsymbol{\eta}\boldsymbol{\cdot}\rho\boldsymbol{v})\,[\overline{\boldsymbol{v}}\,,\,\boldsymbol{\zeta}]\,, \tag{B3b}$$

$$\mathbf{S}^{\Pi} \,=\, -\boldsymbol{\zeta}\,\nabla\boldsymbol{\cdot}(\overline{\rho}\,\Pi\boldsymbol{\eta}) + \boldsymbol{\eta}\,\nabla\boldsymbol{\cdot}(\overline{\rho}\,\Pi\boldsymbol{\zeta}) + \overline{\rho}\,\Pi\,[\boldsymbol{\zeta}\,,\,\boldsymbol{\eta}]\,. \tag{B3c}$$

The adjoint identity (A3a) is used to show that $\boldsymbol{M}$ vanishes and to simplify the expression for $\mathbf{S}^{v}$. Then the identity (A1b) is employed to show that the divergences of $\mathbf{S}^{v}$ and $\mathbf{S}^{\Pi}$ vanish as well. Consequently each of the extra unwanted terms in (B2) vanish, namely

$$\partial_t\,\boldsymbol{M} \,=\, \mathbf{0}\,, \qquad \nabla\boldsymbol{\cdot}(\mathbf{S}^{v} + \mathbf{S}^{\Pi}) \,=\, \mathbf{0}\,. \tag{B4a, b}$$

Hence (B2) and (B4a, b) establish that δL defined by (3.37a) is identical to the elementary form (2.25a), albeit with $\rho\boldsymbol{\mathscr{E}}$ defined by (3.43) rather than (2.25b).

Our final task is to evaluate (3.43) explicitly using (3.37b–d). The result is

$$\rho\boldsymbol{\mathscr{E}} \,=\, \partial_t(\rho\boldsymbol{v}) + \langle(\overline{\boldsymbol{v}} + \boldsymbol{v}')\,,\,\rho\boldsymbol{v}\rangle + (\overline{\rho} + \rho')\,\nabla\Pi + \boldsymbol{\mathscr{S}}^{v} + \boldsymbol{\mathscr{S}}^{\Pi} \tag{B5}$$

similar to (2.25b) but with the additional terms

$$\boldsymbol{\mathscr{S}}^{v} \,=\, -\langle[\overline{\boldsymbol{v}}\,,\,\boldsymbol{\zeta}]\,,\,\rho\boldsymbol{v}\rangle + \langle\overline{\boldsymbol{v}}\,,\,\langle\boldsymbol{\zeta}\,,\,\rho\boldsymbol{v}\rangle\rangle - \langle\boldsymbol{\zeta}\,,\,\langle\overline{\boldsymbol{v}}\,,\,\rho\boldsymbol{v}\rangle\rangle, \tag{B6a}$$

$$\boldsymbol{\mathscr{S}}^{\Pi} \,=\, -\big\langle\boldsymbol{\zeta}\,,\,\overline{\rho}\,\nabla\Pi\big\rangle + \overline{\rho}\,\nabla(\boldsymbol{\zeta}\boldsymbol{\cdot}\nabla\Pi) + \big(\nabla\boldsymbol{\cdot}(\overline{\rho}\boldsymbol{\zeta})\big)\,\nabla\Pi\,. \tag{B6b}$$

Nevertheless use of (A3b) shows that $\boldsymbol{\mathscr{S}}^{v}$ vanishes, while the direct evaluation of $\boldsymbol{\mathscr{S}}^{\Pi}$ shows that it vanishes too giving

$$\boldsymbol{\mathscr{S}}^{v} + \boldsymbol{\mathscr{S}}^{\Pi} \,=\, \mathbf{0}\,. \tag{B6c}$$

Accordingly, together (B5) and (B6c) establish that $\rho\boldsymbol{\mathscr{E}}$ defined by (3.43) is identical to the primitive definition (2.25b).

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